MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

Claude LeBrun
Stony Brook University

April 2, 2020

If $V \in \mathfrak{X}(M)$, we say that

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$,*

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

with $\gamma(0) = p$,

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

with $\gamma(0) = p$, for some $\varepsilon > 0$.

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

with $\gamma(0) = p$, for some $\varepsilon > 0$. Moreover, if V is compactly supported*,*

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

with $\gamma(0) = p$, for some $\varepsilon > 0$. Moreover, if V is compactly supported, *one can take $\varepsilon = \infty$:*

If $V \in \mathfrak{X}(M)$, we say that

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

is an *integral curve* of V if

$$\frac{d}{dt}\gamma(t) = V|_{\gamma(t)}$$

for all $t \in (-\varepsilon, \varepsilon)$.

Theorem. *Given any $V \in \mathfrak{X}(M)$ and any $p \in M$, there exists a unique integral curve*

$$\gamma : (-\varepsilon, \varepsilon) \to M$$

with $\gamma(0) = p$, for some $\varepsilon > 0$. Moreover, if V is compactly supported*, one can take $\varepsilon = \infty$:*

$$\gamma : \mathbb{R} \to M.$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported*

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact.

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp } V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp } V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp}\, V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp } V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp } V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\text{supp}\, V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\Longleftrightarrow$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

We can also define a smooth map

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

We can also define a smooth map

$$\Phi : M \times \mathbb{R} \to M$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\operatorname{supp} V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\implies$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

We can also define a smooth map

$$\Phi : M \times \mathbb{R} \to M$$

by setting

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\mathrm{supp}\ V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\implies$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

We can also define a smooth map

$$\Phi : M \times \mathbb{R} \to M$$

by setting

$$\Phi(p, t) = \Phi_t(p).$$

Recall that $V \in \mathfrak{X}(M)$ *compactly supported* $\iff$

$$\text{supp } V := \overline{\{p \in M \mid V(p) \neq 0\}}$$

is compact. Automatic if M is compact manifold!

$\Longrightarrow$ every integral curve can be extended to

$$\gamma : \mathbb{R} \to M$$

For each $t \in \mathbb{R}$, can then define a smooth map

$$\Phi_t : M \to M$$

by setting

$\Phi_t(p) = \gamma_p(t)$ where γ_p unique integral curve with $\gamma_p(0) = p$.

We can also define a smooth map

$$\Phi : M \times \mathbb{R} \to M$$

by setting

$$\Phi(p, t) = \Phi_t(p).$$

This map is called the *flow* of V.

Theorem.

Theorem. *Given any* compactly supported

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$,

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$*, the maps*

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$*, the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms,

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms, and satisfy

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms, and satisfy

$$\Phi_0 = id_M$$

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms, and satisfy

$$\Phi_0 = id_M$$

$$\Phi_u \circ \Phi_t = \Phi_{u+t}.$$

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms, and satisfy

$$\Phi_0 = id_M$$

$$\Phi_u \circ \Phi_t = \Phi_{u+t}.$$

Thus, these form a one-parameter group of diffeomorphisms $M \to M$.

Theorem. *Given any* compactly supported $V \in \mathfrak{X}(M)$, *the maps*

$$\Phi_t : M \to M$$

are all diffeomorphisms, and satisfy

$$\Phi_0 = id_M$$

$$\Phi_u \circ \Phi_t = \Phi_{u+t}.$$

Thus, these form a one-parameter group of diffeomorphisms $M \to M$*.*

Notice that

$$\Phi_{-t} = (\Phi_t)^{-1}$$

When V not compactly supported, "flow" only defined on neighborhood of $M \times \{0\} \subset M \times \mathbb{R}$:

$$\Phi : M \times \mathbb{R} \dashrightarrow M$$

where dashed arrow means "not defined everywhere."

Theorem. *Let* M *be a smooth* n*-manifold, let* $V \in \mathfrak{X}(M)$ *be a smooth vector field, and let* $p \in M$ *be a point where* $V(p) \neq 0$.

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$*

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p*

Theorem. *Let* M *be a smooth* n*-manifold, let* $V \in \mathfrak{X}(M)$ *be a smooth vector field, and let* $p \in M$ *be a point where* $V(p) \neq 0$*. Then the exists a smooth coordinate chart* $(x^1, \ldots, x^n)$ *defined on a neighborhood* $\mathscr{U} \subset M$ *of* p *in which*

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq 0$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Remark. Of course, since $\frac{\partial}{\partial x^1} \neq \mathsf{0}$ everywhere, this can actually be done if and only if $V(p) \neq \mathsf{0}$!

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq 0$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Restrict flow $\Phi : M \times \mathbb{R} \dashrightarrow M$ of V to $N \times \mathbb{R}$

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Restrict flow $\Phi : M \times \mathbb{R} \dashrightarrow M$ of V to $N \times \mathbb{R}$ to define smooth map

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Restrict flow $\Phi : M \times \mathbb{R} \dashrightarrow M$ of V to $N \times \mathbb{R}$ to define smooth map

$$F : N \times \mathbb{R} \dashrightarrow M.$$

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Restrict flow $\Phi : M \times \mathbb{R} \dashrightarrow M$ of V to $N \times \mathbb{R}$ to define smooth map

$$F : N \times \mathbb{R} \dashrightarrow M.$$

Inverse function theorem: local diffeo near p.

Theorem. *Let M be a smooth n-manifold, let $V \in \mathfrak{X}(M)$ be a smooth vector field, and let $p \in M$ be a point where $V(p) \neq \mathsf{0}$. Then the exists a smooth coordinate chart $(x^1, \ldots, x^n)$ defined on a neighborhood $\mathscr{U} \subset M$ of p in which*

$$V = \frac{\partial}{\partial x^1}.$$

Proof. Let $N \subset M$ be an embedded $(n-1)$-dim'l submanifold through p such that $V(p) \notin T_pN$.

Restrict flow $\Phi : M \times \mathbb{R} \dashrightarrow M$ of V to $N \times \mathbb{R}$ to define smooth map

$$F : N \times \mathbb{R} \dashrightarrow M.$$

Inverse function theorem: local diffeo near p. Introduce local coordinates $(x^2, \ldots, x^n)$ on N, set $x^1 = t$ on $\mathbb{R}$, pull back to $\mathscr{U} \subset M$ via F^{-1}. □

Lie derivative:

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

We then get a family of "pulled back" vector fields, depending on $t \in \mathbb{R}$, defined as

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

We then get a family of "pulled back" vector fields, depending on $t \in \mathbb{R}$, defined as

$$\Phi_t^* W := (\Phi_t^{-1})_* W = (\Phi_{-t})_* W.$$

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

We then get a family of "pulled back" vector fields, depending on $t \in \mathbb{R}$, defined as

$$\Phi_t^* W := (\Phi_t^{-1})_* W = (\Phi_{-t})_* W.$$

At each point, this gives us a family of tangent vectors, depending differentiably on t.

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

We then get a family of "pulled back" vector fields, depending on $t \in \mathbb{R}$, defined as

$$\Phi_t^* W := (\Phi_t^{-1})_* W = (\Phi_{-t})_* W.$$

At each point, this gives us a family of tangent vectors, depending differentiably on t.

We can therefore define the Lie derivative of W with respect to V to be

Lie derivative:

Let V be a smooth vector field, with smooth flow

$$\{\Phi_t : M \dashrightarrow M\}.$$

Now suppose W is another vector field.

We then get a family of "pulled back" vector fields, depending on $t \in \mathbb{R}$, defined as

$$\Phi_t^* W := (\Phi_t^{-1})_* W = (\Phi_{-t})_* W.$$

At each point, this gives us a family of tangent vectors, depending differentiably on t.

We can therefore define the Lie derivative of W with respect to V to be

$$\mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0}.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Very different-looking definitions!

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Very different-looking definitions!

Lie derivative:

$$\mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0}.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Very different-looking definitions!

Lie derivative:

$$\mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0}.$$

Lie bracket:

$$[V, W]f = V(Wf) - W(Vf).$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. First consider the open subset

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. First consider the open subset

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M.$$

Near any $p \in \mathcal{A}$, $\exists$ coordinates $(x^1, \ldots, x^n)$ s.t.

$$V = \frac{\partial}{\partial x^1}.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. First consider the open subset

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M.$$

Near any $p \in \mathcal{A}$, $\exists$ coordinates $(x^1, \ldots, x^n)$ s.t.

$$V = \frac{\partial}{\partial x^1}.$$

Flow Φ_t in these coordinates is

$$(x^1, x^2, \ldots, x^n) \longmapsto (x^1 + t, x^2, \ldots, x^n).$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. First consider the open subset

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M.$$

Near any $p \in \mathcal{A}$, $\exists$ coordinates $(x^1, \dots, x^n)$ s.t.

$$V = \frac{\partial}{\partial x^1}.$$

Flow Φ_t in these coordinates is

$$(x^1, x^2, \dots, x^n) \longmapsto (x^1 + t, x^2, \dots, x^n).$$

So

$$\Phi_t^* W = \sum W^j (x^1 + t, x^2, \dots, x^n) \frac{\partial}{\partial x^j}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\mathcal{L}_V W \ := \ \frac{d}{dt}(\Phi_t^* W)\bigg|_{t=0}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned} \mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\sum W^j(x^1 + t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Big|_{t=0} \end{aligned}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned}
\mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sum W^j(x^1 + t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Bigg|_{t=0} \\
&= \sum \frac{\partial W^j}{\partial x^1}(x^1, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}
\end{aligned}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned}
\mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sum W^j(x^1 + t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Bigg|_{t=0} \\
&= \sum \frac{\partial W^j}{\partial x^1}(x^1, x^2, \ldots, x^n)\frac{\partial}{\partial x^j} \\
&= \left[\frac{\partial}{\partial x^1}, \sum W^j \frac{\partial}{\partial x^j}\right]
\end{aligned}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned}
\mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sum W^j(x^1 + t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Bigg|_{t=0} \\
&= \sum \frac{\partial W^j}{\partial x^1}(x^1, x^2, \ldots, x^n)\frac{\partial}{\partial x^j} \\
&= \left[\frac{\partial}{\partial x^1}, \sum W^j \frac{\partial}{\partial x^j}\right] \\
&= [V, W]
\end{aligned}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned}
\mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sum W^j(x^1 + t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Big|_{t=0} \\
&= \sum \frac{\partial W^j}{\partial x^1}(x^1, x^2, \ldots, x^n)\frac{\partial}{\partial x^j} \\
&= \left[\frac{\partial}{\partial x^1}, \sum W^j \frac{\partial}{\partial x^j}\right] \\
&= [V, W]
\end{aligned}$$

So $\mathcal{L}_V W = [V, W]$ on

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. So, in these coordinates,

$$\begin{aligned}
\mathcal{L}_V W &:= \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\sum W^j(x^1+t, x^2, \ldots, x^n)\frac{\partial}{\partial x^j}\right)\Big|_{t=0} \\
&= \sum \frac{\partial W^j}{\partial x^1}(x^1, x^2, \ldots, x^n)\frac{\partial}{\partial x^j} \\
&= \left[\frac{\partial}{\partial x^1}, \sum W^j \frac{\partial}{\partial x^j}\right] \\
&= [V, W]
\end{aligned}$$

So $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. Now $\mathcal{L}_V W$ and $[V, W]$ are both smooth vector fields.

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. Now $\mathcal{L}_V W$ and $[V, W]$ are both smooth vector fields. So the set of $p \in M$ where

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. Now $\mathcal{L}_V W$ and $[V, W]$ are both smooth vector fields. So the set of $p \in M$ where

$$\mathcal{L}_V W - [V, W] = 0$$

is closed.

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. Now $\mathcal{L}_V W$ and $[V, W]$ are both smooth vector fields. So the set of $p \in M$ where

$$\mathcal{L}_V W = [V, W]$$

is closed.

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M,$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv 0 \text{ near } p\}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv \mathsf{0} \text{ near } p\} = \text{Int}\,\{p \mid V(p) = \mathsf{0}\}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv \mathsf{0} \text{ near } p\} = \text{Int } \{p \mid V(p) = \mathsf{0}\}$$

then flow is trivial on $\mathcal{B}$,

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv 0 \text{ near } p\} = \operatorname{Int} \{p \mid V(p) = 0\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = 0 \qquad\qquad \text{on } \mathcal{B}.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv \mathsf{0} \text{ near } p\} = \text{Int } \{p \mid V(p) = \mathsf{0}\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = \mathsf{0} = [V, W] \quad \text{on } \mathcal{B}.$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv 0 \text{ near } p\} = \operatorname{Int}\{p \mid V(p) = 0\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = 0 = [V, W] \quad \text{on } \mathcal{B}.$$

$$M = \overline{\mathcal{A}} \cup \mathcal{B}$$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq \mathsf{0}\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv \mathsf{0} \text{ near } p\} = \text{Int}\, \{p \mid V(p) = \mathsf{0}\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = \mathsf{0} = [V, W] \quad \text{on } \mathcal{B}.$$

$M = \overline{\mathcal{A}} \cup \mathcal{B} \Longrightarrow$

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$*,*

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv 0 \text{ near } p\} = \text{Int}\,\{p \mid V(p) = 0\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = 0 = [V, W] \quad \text{on } \mathcal{B}.$$

$M = \overline{\mathcal{A}} \cup \mathcal{B} \Longrightarrow \mathcal{L}_V W = [V, W]$ everywhere.

Theorem. *For any vector fields* $V, W \in \mathfrak{X}(M)$,

$$\mathcal{L}_V W = [V, W].$$

Proof. But since $\mathcal{L}_V W = [V, W]$ on

$$\mathcal{A} := \{p \mid V(p) \neq 0\} \subset M,$$

we therefore also have

$$\mathcal{L}_V W = [V, W] \quad \text{on} \quad \overline{\mathcal{A}}$$

by continuity.

But if

$$\mathcal{B} := \{p \mid V(p) \equiv 0 \text{ near } p\} = \text{Int}\,\{p \mid V(p) = 0\}$$

then flow is trivial on $\mathcal{B}$, and hence

$$\mathcal{L}_V W = 0 = [V, W] \quad \text{on } \mathcal{B}.$$

$M = \overline{\mathcal{A}} \cup \mathcal{B} \Longrightarrow \mathcal{L}_V W = [V, W]$ everywhere. $\square$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$*, and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow$$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \iff \Phi_t^* W = W \quad \forall t.$$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$*, and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \iff \Phi_t^* W = W \quad \forall t.$$

Proof.

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t^* W = W \quad \forall t.$$

Proof.

$$\mathcal{L}_V W = [V, W],$$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t^* W = W \quad \forall t.$$

Proof.

$$\mathcal{L}_V W = [V, W],$$

and, because $\Phi_u \circ \Phi_t = \Phi_{u+t}$,

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t^* W = W \quad \forall t.$$

Proof.

$$\mathcal{L}_V W = [V, W],$$

and, because $\Phi_u \circ \Phi_t = \Phi_{u+t}$,

$$\mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \Longrightarrow$$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and let*

$$\Phi_t = \textit{flow of } V.$$

Then

$$[V, W] = 0 \iff \Phi_t^* W = W \quad \forall t.$$

Proof.

$$\mathcal{L}_V W = [V, W],$$

and, because $\Phi_u \circ \Phi_t = \Phi_{u+t}$,

$$\mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=0} \Longrightarrow$$

$$\Phi_u^* \mathcal{L}_V W := \frac{d}{dt}(\Phi_t^* W)\Big|_{t=u}.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$*, and set*

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeomorphisms

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeo-morphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeo-morphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Then $\Pi_0 = \mathrm{id}_M$, and, for any p,

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeomorphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Then $\Pi_0 = \mathrm{id}_M$, and, for any p,

$$\frac{d}{du}\Pi_u(p) = \Phi_t^* W|_{\Pi_u(p)}\,.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$*, and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \iff \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeomorphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Then $\Pi_0 = \mathrm{id}_M$, and, for any p,

$$\frac{d}{du}\Pi_u(p) = \Phi_t^* W|_{\Pi_u(p)}.$$

Thus, Π_u is flow of $\Phi_t^* W$, and

Theorem. *Let* $V, W \in \mathfrak{X}(M)$*, and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeomorphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Then $\Pi_0 = \mathrm{id}_M$, and, for any p,

$$\frac{d}{du}\Pi_u(p) = \Phi_t^* W|_{\Pi_u(p)}.$$

Thus, Π_u is flow of $\Phi_t^* W$, and

$$\Phi_t^* W = W \quad \Longleftrightarrow \quad \Phi_t^{-1} \circ \Psi_u \circ \Phi_t = \Psi_u \quad \forall u.$$

Theorem. *Let* $V, W \in \mathfrak{X}(M)$, *and set*

$$\Phi_t = \textit{flow of } V, \quad \textit{and} \quad \Psi_u = \textit{flow of } W.$$

Then

$$[V, W] = 0 \quad \Longleftrightarrow \quad \Phi_t \circ \Psi_u = \Psi_u \circ \Phi_t \quad \forall u, t.$$

Proof. For a fixed value of t, the family of diffeomorphisms

$$\Pi_u := \Phi_t^{-1} \circ \Psi_u \circ \Phi_t.$$

Then $\Pi_0 = \mathrm{id}_M$, and, for any p,

$$\frac{d}{du}\Pi_u(p) = \Phi_t^* W|_{\Pi_u(p)}.$$

Thus, Π_u is flow of $\Phi_t^* W$, and

$$\Phi_t^* W = W \quad \Longleftrightarrow \quad \Phi_t^{-1} \circ \Psi_u \circ \Phi_t = \Psi_u \quad \forall u.$$

However, $\Phi_t^* W = W \ \forall t \Longleftrightarrow [V, W] = 0.$ $\square$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$,

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and suppose that, the vectors* $V(p)$ *and* $W(p)$ *are linearly independent in* T_pM.

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and suppose that, for some* $p \in M$, *the vectors* $V(p)$ *and* $W(p)$ *are linearly independent in* $T_p M$.

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and suppose that, for some* $p \in M$, *the vectors* $V(p)$ *and* $W(p)$ *are linearly independent in* T_pM. *Then* $[V, W] = 0$ *on some neighborhood of* p $\iff$

Corollary. *Let* $V, W \in \mathfrak{X}(M)$, *and suppose that, for some* $p \in M$, *the vectors* $V(p)$ *and* $W(p)$ *are linearly independent in* T_pM. *Then* $[V, W] = 0$ *on some neighborhood of* $p \iff \exists$ *coordinates* $(x^1, \ldots, x^n)$ *on a neighborhood of* p *in which*

Corollary. *Let* $V, W \in \mathfrak{X}(M)$*, and suppose that, for some* $p \in M$*, the vectors* $V(p)$ *and* $W(p)$ *are linearly independent in* T_pM*. Then* $[V, W] = 0$ *on some neighborhood of* $p \iff \exists$ *coordinates* $(x^1, \ldots, x^n)$ *on a neighborhood of* p *in which*

$$V = \frac{\partial}{\partial x^1} \quad \textit{and} \quad W = \frac{\partial}{\partial x^2}.$$