MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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are all linear maps $\mathbb{V} \to \mathbb{R}$, for any fixed vectors $\mathsf{V}_1, \mathsf{V}_2, \ldots, \mathsf{V}_k \in \mathbb{V}$.

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Then ϕ is "bilinear" (i.e. multilinear, with $k = 2$).

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called a simple tensor product.

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For us, the most important case will be $\bigotimes^k \mathbb{V}^*$.

If $\mathsf{V} \in \mathbb{V}$, and $\phi \in \bigotimes^k \mathbb{V}^*$, then we may define

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Pronounced: "V contract ϕ" or "V hook ϕ"

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Similarly, for example,

$$\mathbb{V}^* \otimes \mathbb{V} \cong \mathrm{Hom}(\mathbb{V}, \mathbb{V}) := \mathrm{End}(\mathbb{V}).$$

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(This module structure actually contains enough information to reconstruct the bundle E, but we will never explictly need this fact in our course.)

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