MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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Good generalization for vector fields that are C^k.

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This makes $\mathfrak{X}(M)$ into a "Lie algebra."

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Notice that

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General k similar; proceed by induction.