Final Exam

Geometry/Topology II

Spring 2020

This is an open-book exam, but collaboration is forbidden.

Your exam paper will be due at 5:00 pm on Tuesday, May 19, 2020.

Do exactly **five** problems. Each problem is worth 20 points.

Please e-mail your solutions to Prof. LeBrun, either as

- a scan of your handwritten solutions, or as
- a PDF of your solutions that you have typeset with TeX or LaTeX.

Your submission must include a signed copy of the following statement:

On my honor, I have neither given nor received aid on this exam.

On this exam, *manifold* always means "manifold without boundary," except when the hyphenated term "manifold-with-boundary" is used.

The notation H^k is used on this exam only in reference to *de Rham* cohomology.

1. Let M and N be smooth manifolds. Show that $M \times N$ is orientable $\iff$ M and N are both orientable.

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2. Let M be a smooth m-manifold, and suppose that ω is a smooth m-form which is non-zero at every point of M. Show that every point of M has a neighborhood on which there exist coordinates $(x^1, x^2, \ldots, x^m)$ in which

$$\omega = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^m.$$

Then use this to prove that a smooth manifold is orientable iff it admits a smooth atlas whose coordinate transition maps $(x^1, \ldots, x^m) \mapsto (y^1, \ldots, y^m)$ all satisfy

$$\det \left[\frac{\partial y^j}{\partial x^k} \right] \equiv 1.$$

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3. Let ψ be the 2-form on $\mathbb{R}^3 - \{\mathsf{0}\}$ defined by

$$\psi = \frac{x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let $\Sigma \subset \mathbb{R}^3 - \{\mathsf{0}\}$ be a smooth compact surface that is the boundary $\partial \mathcal{U}$ of a compact 3-manifold-with-boundary $\mathcal{U} \subset \mathbb{R}^3$. Let's agree to give the "bounded domain" $\mathcal{U}$ the orientation it inherits from $\mathbb{R}^3$, and then use this to induce the corresponding "out-pointing" boundary orientation on $\Sigma = \partial \mathcal{U}$. Prove that

$$\frac{1}{4\pi} \int_\Sigma \psi = \begin{cases} 1 & \text{if } \mathsf{0} \in \mathcal{U}, \\ 0 & \text{otherwise.} \end{cases}$$

4. Let m and n be positive integers. Show that there is a degree-1 smooth map $S^n \times S^m \to S^{n+m}$, but that any any smooth map $S^{n+m} \to S^n \times S^m$ has degree zero.

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5. Let M be a smooth compact connected manifold, and suppose that $\varphi \in \Omega^1(M)$ is a closed 1-form that satisfies $\varphi \neq 0$ at every point of M. Show that $H^1(M) \neq 0$, that $\pi_1(M)$ is infinite, and that the universal cover $\widetilde{M}$ of M is non-compact.

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6. Consider the **two** vector fields

$$V = \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} \qquad \text{and} \qquad W = \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}$$

on $\mathbb{R}^4$. Is there is a **3**-dimensional submanifold $X^3 \subset \mathbb{R}^4$ that passes through the origin $(0,0,0,0)$ and is tangent to both V and W at every $p \in X$? If so, find it explicitly. Otherwise, prove that it cannot exist.

7. Let M be a smooth m-manifold, $m \geq 2$, and let $\omega \in \Omega^m(M)$ be a top-degree form that satisfies $\omega \neq 0$ at every point. Such a differential form ω is often called a "volume form," because we can use it to assign an m-dimensional volume to any open set in M via integration. We now fix a specific volume form ω for the remainder of this problem.

For any $(m-2)$-form $\varphi \in \Omega^{m-2}(M)$, show that the equation

$$V \lrcorner\, \omega = d\varphi \tag{1}$$

defines a unique vector field $V \in \mathfrak{X}(M)$, and that this vector field is then "volume preserving," in the sense that ω is invariant under its flow. What cohomological condition on M is equivalent to the statement that every "volume preserving" vector field arises from some φ via equation (1)?

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8. Let

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \approx \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$$

be the so-called n-dimensional torus, or *n-torus*. Show that each of the constant-coefficient differential forms

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Omega^k(\mathbb{R}^n), \qquad 1 \leq i_1 < \cdots < i_k \leq n,$$

is the pull-back to $\mathbb{R}^n$ of a closed differential form on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Using an integration argument, then prove that the cohomology classes of these forms are linearly independent in $H^k(\mathbb{T}^n)$. As a consequence, conclude that

$$\dim H^k(\mathbb{T}^n) \geq \binom{n}{k}$$

for any integers k and n with $0 \leq k \leq n$.

9. If M is a smooth n-manifold such that $\dim H^k(M)$ is finite-dimensional for each k, the *Euler characteristic* of M is defined to be the integer

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M).$$

If $M = U \cup V$, where $U, V \subset M$ are open sets such that $H^k(U)$, $H^k(V)$, and $H^k(U \cap V)$ are finite-dimensional for every k, show that

$$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V).$$

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10. (Only for students who have done Problem 9.) Let M be a smooth manifold for which $\dim H^k(M)$ is finite-dimensional for each k. Notice that $M \times S^n$ can be written as the union of two open sets, each of which is diffeomorphic to $M \times \mathbb{R}^n$. Using this observation and the result proved in Problem 9, prove, by induction on n, that

$$\chi(M \times S^n) = \begin{cases} 2\chi(M) & \text{if } n \text{ is even,} \\ 0 & \text{if n is odd.} \end{cases}$$

Then use this to calculate the Euler characteristic of any Cartesian product

$$S^{n_1} \times S^{n_2} \times \cdots \times S^{n_\ell}$$

of a finite collection of spheres.