MAT 531

Geometry/Topology II

Introduction to Smooth Manifolds

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are all linear maps $\mathbb{V} \to \mathbb{R}$, for any fixed vectors $V_1, V_2, \ldots, V_k \in \mathbb{V}$.

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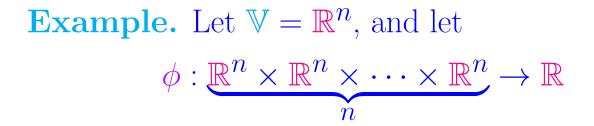
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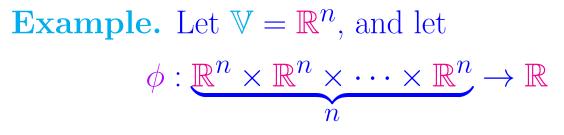
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Then $\alpha \otimes \beta \otimes \cdots \otimes \gamma$ is a multilinear map, called a simple tensor product.

The k^{th} tensor product of the dual vector space

$$\otimes^k \mathbb{V}^* = \underbrace{\mathbb{V}^* \otimes \mathbb{V}^* \otimes \cdots \otimes \mathbb{V}^*}_k$$

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Of course,

$$\otimes^1 \mathbb{V}^* = \mathbb{V}^*.$$

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By convention,

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Elements of all such spaces are said to be tensors on \mathbb{V} .

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For us, the most important case will be $\otimes^k \mathbb{V}^*$.

Pronounced: "V contract ϕ " or "V hook ϕ "

by

$$(\mathsf{V} \lrcorner \phi)(\mathsf{V}_1, \ldots, \mathsf{V}_{k-1}) = \phi(\mathsf{V}, \mathsf{V}_1, \ldots, \mathsf{V}_{k-1}).$$

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This gives rise to an isomorphism $\otimes^{k} \mathbb{V}^{*} \cong \operatorname{Hom}(\mathbb{V}, \otimes^{k-1} \mathbb{V}^{*}).$

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where Hom means "linear maps between..."

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 \implies If dim $\mathbb{V} \geq 2$, most elements of $\otimes^k \mathbb{V}^*$ are not simple tensor products!

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Indeed, a simple tensor product $\alpha \otimes \beta \cdots$ gives rise to a homomorphism

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Similarly, for example,

 $\mathbb{V}^* \otimes \mathbb{V} \cong \operatorname{Hom}(\mathbb{V}, \mathbb{V}) := \operatorname{End}(\mathbb{V}).$

If M is a smooth n-manifold, and if $p \in M$,

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associated with a coordinate system (x^1, \ldots, x^n) on \mathscr{U} .

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$$TM|_{\mathscr{U}} = T\mathscr{U} \cong \mathscr{U} \times \mathbb{R}^n$$

by using the coordinate basis

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(This module structure actually contains enough information to reconstruct the bundle E, but we will never explicitly need this fact in our course.) Important case for us:

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that is multilinear over $C^{\infty}(M)$ arises from a unique tensor field $\varphi \in \Gamma(\otimes^k T^*M)$.

Example. Let $\mathbf{u} \in C^{\infty}(M)$.

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Pull-backs.

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Thus, $M \rightsquigarrow \Gamma(\otimes^k T^*M)$ is a contravariant functor.

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covariant tensor fields. Most of us now view this as old-fashioned, deprecated terminology that belongs on the trash-heap of history. Lee, unfortunately, does not agree. So, even though I will avoid using this misleading terminology myself, you will need to understand what he means by it in order to do some of the homework problems.