Mid-Term Solutions

Geometry/Topology II

Spring 2015

Do four of the following problems. 25 points each.

The term manifold is used on this exam to mean a manifold *without* boundary.

1. Let N be a smooth n-manifold, and let $L \subset N$ and $M \subset N$ be a smoothly embedded submanifolds of dimensions ℓ and m, respectively. One says that L and M are *transverse* if, at every $p \in L \cap M$, the tangent space of L and M together span the tangent space of N:

$$T_pL + T_pM = T_pN.$$

If L and M are transverse, prove that $L \cap M$ is a smoothly embedded submanifold of N. What is the dimension of $L \cap M$?

Given a point $p \in L \subset N$, we can find smooth coordinates $(x^1, \ldots, x^n)$ on a neighborhood $\mathcal{U} \subset N$ of p in which $L \cap \mathcal{U}$ becomes $\{(x^1, \ldots x^\ell, 0, \ldots, 0)\}$. In other words,

$$\begin{array}{ccc} \mathcal{U} & \stackrel{F}{\longrightarrow} & \mathbb{R}^{n-\ell} \\ (x^1, \ldots x^\ell, x^{\ell+1}, \ldots x^n) & \longmapsto & (x^{\ell+1}, \ldots x^n) \end{array}$$

is a submersion such that $F^{-1}(\mathbf{0}) = L \cap \mathcal{U}$.

Now suppose that $p \in L \cap M$, and that $T_pL + T_pM = T_pN$. Since $\ker dF_p = T_pL$, it follows that

$$dF_p(T_pM) = dF_p(T_pN) = T_{\mathbf{0}}\mathbb{R}^n = \mathbb{R}^{n-\ell}.$$

The map

$$f := F|_{\mathcal{U}\cap M} : \mathcal{U} \cap M \to \mathbb{R}^{n-\ell}$$

therefore has maximal rank at p, and therefore has maximal rank on a neighborhood $\mathcal{V} \subset M$ of $p \in M$. This means that

$$\tilde{f} := f|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}^{n-\ell}$$

is a submersion, and an open set of p in $L \cap M$ therefore coincides with the submanifold $\tilde{f}^{-1}(\mathbf{0})$. If L and M are transverse, this shows that $L \cap M$ is an embedded submanifold of M, and that its dimension is $m-(n-\ell) = \ell+m-n$. Moreover, since $L \cap M$ is a smoothly embedded submanifold of M, and M is a smoothly embedded submanifold of N, it follows that $L \cap M$ is also a smoothly embedded submanifold of N.

2. Show that there does not exist an immersion $F : T^2 \to S^2$ from the 2-torus to the 2-sphere. (**Hint:** First prove that such an immersion would have to be a covering map.)

If $F : M^n \to N^n$ is a smooth immersion between manifolds of the same dimension, it is necessarily a local diffeomorphism by the inverse function theorem. Now suppose that M is compact, and let $q \in N$. The pre-image $F^{-1}(q)$ of q is then compact and discrete, and therefore is a finite set $\{p_1, \dots p_k\}$; and moreover, each p_j has a neighborhood $\mathcal{U}_j$ which is mapped diffeomorphically to some neighborhood $\mathcal{V}_j$ of q. Since M is Hausdorff, we can, by induction, also assume that these open sets $\mathcal{U}_j$ are mutually disjoint. Now since M is compact, $\mathcal{W} = N - F(M - \cup_j \mathcal{U}_j)$ is the complement of a compact set, and hence open, because N is Hausdorff; and, since every pre-image p_j of q belongs to $\cup_j \mathcal{U}_j$, the open set $\mathcal{W}$ contains q. If we now set $\mathcal{V} := \mathcal{V}_1 \cap \cdots \cap \mathcal{V}_k \cap \mathcal{W}$, then the pre-image of any $\tilde{q} \in \mathcal{V}$ is a subset of $\cup_j \mathcal{U}_j$, and it therefore follows that $F^{-1}(\mathcal{V})$ is the union of k disjoint open sets $F^{-1}(\mathcal{V}) \cap \mathcal{U}_j$, each of which is mapped diffeomorphically to $\mathcal{V}$ by F. This shows that N is evenly covered by F, and it follows that F is a covering map if M and N are also both assumed to be (path-wise) connected.

Since T^2 and S^2 are smooth compact connected manifolds of the same dimension, it follows that any immersion $F : T^2 \to S^2$ would have to be a covering map. In particular, the induced map $F_\# : \pi_1(T^2) \to \pi_1(S^2)$ would have to be injective. But $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, whereas $\pi_1(S^2) = 0$, so this is a contradiction. This shows that a smooth map $F : T^2 \to S^2$ can never be an immersion.

3. Consider the vector fields V and W on $\mathbb{R}^3$ defined by

$$\begin{aligned} V &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\ W &= \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \end{aligned}$$

in the standard coordinate system $(x^1, x^2, x^3) = (x, y, z)$.

(a) Explicitly find the flow generated by V.

(b) Compute the Lie derivative $\mathcal{L}_V W$ *directly from the definition.*

(c) Compute the Lie bracket $[V, W]$ *directly from the definition.*

(d) How are your answers to (b) and (c) related? Explain.

(a) The flow of V is obtained by solving the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= -y \\ \frac{dz}{dt} &= z \end{aligned}$$

which "decouple," insofar as all three can be solved separately:

$$\begin{aligned} x(t) &= e^t x(0) \\ y(t) &= e^{-t} y(0) \\ z(t) &= e^t z(0) \end{aligned}$$

Thus, the flow of V is explicitly given by

$$\Phi_t(x, y, z) = (e^t x, e^{-t} y, e^t z).$$

(b) Recall that

$$\mathcal{L}_V W := \frac{d}{dt}[\Phi_{(-t)*}W]\Big|_{t=0}.$$

On the other hand,

$$\Phi_{-t}(x,y,z) = (e^{-t}x, e^t y, e^{-t}z)$$

is a linear map for each t, with differential given by the Jacobian matrix

$$d\Phi_{-t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

relative to the standard basis $\partial/\partial x, \partial/\partial y, \partial/\partial z$. Thus

$$\Phi_{(-t)*}W = \Phi_{(-t)*}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) = e^{-t}\frac{\partial}{\partial x} + e^t\frac{\partial}{\partial y} + e^{-t}\frac{\partial}{\partial z},$$

and

$$\frac{d}{dt}[\Phi_{(-t)*}W] = -e^{-t}\frac{\partial}{\partial x} + e^t\frac{\partial}{\partial y} - e^{-t}\frac{\partial}{\partial z}.$$

Hence

$$\mathcal{L}_V W = \left(-e^{-t}\frac{\partial}{\partial x} + e^t\frac{\partial}{\partial y} - e^{-t}\frac{\partial}{\partial z}\right)\Big|_{t=0} = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

(c) By definition,

$$\begin{aligned}
[V,W]f &= V(Wf) - W(Vf) \\
&= \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f \\
&\qquad - \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) f \\
&= -\left(\frac{\partial x}{\partial x}\frac{\partial}{\partial x} - \frac{\partial y}{\partial y}\frac{\partial}{\partial y} + \frac{\partial z}{\partial z}\frac{\partial}{\partial z}\right) f \\
&= -\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f
\end{aligned}$$

So

$$[V,W] = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

(d) The answers to (b) and (c) are identical. This illustrates the theorem that $\mathcal{L}_V W = [V,W]$ for any smooth vector fields V and W.

4. Let $A = \left[A^k_j\right]$ and $B = \left[B^k_j\right]$ be $n \times n$ matrices, and use these matrices to define the two vector fields

$$\begin{aligned} X &= \sum_{j,k=1}^{n} A^k_j x^j \frac{\partial}{\partial x^k} \\ Y &= \sum_{j,k=1}^{n} B^k_j x^j \frac{\partial}{\partial x^k} \end{aligned}$$

on $\mathbb{R}^n$. Prove that $[X,Y] = 0 \Leftrightarrow A$ and B commute under matrix multiplication.

$$\begin{aligned} [X,Y] &= \left(\sum_{j,k=1}^{n} A^k_j x^j \frac{\partial}{\partial x^k}\right)\left(\sum_{\ell,m=1}^{n} B^m_\ell x^\ell \frac{\partial}{\partial x^m}\right) \\ &\qquad - \left(\sum_{j,k=1}^{n} B^k_j x^j \frac{\partial}{\partial x^k}\right)\left(\sum_{\ell,m=1}^{n} A^m_\ell x^\ell \frac{\partial}{\partial x^m}\right) \\ &= \sum_{j,k,\ell,m=1}^{n} A^k_j x^j \frac{\partial}{\partial x^k}\left(B^m_\ell x^\ell\right)\frac{\partial}{\partial x^m} - \sum_{j,k,\ell,m=1}^{n} B^k_j x^j \frac{\partial}{\partial x^k}\left(A^m_\ell x^\ell\right)\frac{\partial}{\partial x^m} \\ &= \sum_{j,k,\ell,m=1}^{n} A^k_j x^j B^m_\ell \delta^\ell_k \frac{\partial}{\partial x^m} - \sum_{j,k,\ell,m=1}^{n} B^k_j x^j A^m_\ell \delta^\ell_k \frac{\partial}{\partial x^m} \\ &= \sum_{j,k,m=1}^{n} A^k_j x^j B^m_k \frac{\partial}{\partial x^m} - \sum_{j,k,m=1}^{n} B^k_j x^j A^m_k \frac{\partial}{\partial x^m} \\ &= \sum_{j,k,m=1}^{n} \left(B^m_k A^k_j - A^m_k B^k_j\right) x^j \frac{\partial}{\partial x^m} \end{aligned}$$

Thus $[X,Y]=0$ iff

$$\sum_k A_k^m B_j^k = \sum_k B_k^m A_j^k,$$

and the latter is equivalent to saying that $AB = BA$ as matrices.

5. Let M be a compact m-manifold, and suppose that $F : M \to S^1$ is a submersion from M to the circle. Let $X = \partial/\partial\theta$ be the standard unit vector field on the circle. Show that there exists a vector field V on M such that $(dF)_p(V_p) = X_{F(p)}$ for every $p \in M$. Then use the flow of V to prove that, for any two points $q, \tilde{q} \in S^1$, the compact $(m-1)$-manifolds $F^{-1}(q)$ and $F^{-1}(\tilde{q})$ are diffeomorphic.

Since F is a submersion, we can cover M with coordinate domains $\mathcal{U}_\alpha$ on which we have coordinates $(x_\alpha^1, \ldots, x_\alpha^m)$ in which F takes the form

$$\theta = x_\alpha^1,$$

where θ is a local "angle" coordinate on S^1. The vector field $V_\alpha = \partial/\partial x_\alpha^1$ defined on $\mathcal{U}_\alpha$ therefore has the property that $(dF)(V_\alpha) = \partial/\partial\theta = X$ at every point of $\mathcal{U}_\alpha$. The difficulty, of course, is that the vector fields V_α and V_β will general disagree on their common domains of definition.

To get around this difficulty, we now let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\mathcal{U}_\alpha$ of M, and set

$$V = \sum_\alpha \phi_\alpha V_\alpha.$$

This sum is locally finite, and the $\phi_\alpha V_\alpha$ is understood to mean the smooth vector field on all of M given by

$$(\phi_\alpha V_\alpha|)(p) = \begin{cases} \phi_\alpha(p) V_\alpha(p) & \text{if } p \in \mathcal{U}_\alpha, \\ 0 & \text{if } p \notin \mathcal{U}_\alpha. \end{cases}$$

Since $\sum_\alpha \phi_\alpha \equiv 1$, we therefore have

$$\begin{aligned}
(dF)_p(V_p) &= (dF)_p\left(\sum_\alpha \phi_\alpha(p)V_\alpha(p)\right) \\
&= \sum_\alpha \phi_\alpha(p)(dF)_p(V_\alpha(p)) \\
&= \sum_\alpha \phi_\alpha(p)X_{F(p)} \\
&= \left[\sum_\alpha \phi_\alpha(p)\right] X_{F(p)} \\
&= 1 \cdot X_{F(p)} \\
&= X_{F(p)},
\end{aligned}$$

so V is a smooth vector field on M with the required property.

Since M is compact, V is compactly supported, and its flow $\Phi_t : M \to M$ is defined for all $t \in \mathbb{R}$. Similarly, the flow $\Psi_t : S^1 \to S^1$ of X is defined for all t. But since $(dF)(V) = X$, we must that

$$F \circ \Phi_t = \Psi_t \circ F.$$

In other words, the diffeomorphism $\Phi_t : M \to M$ sends $F^{-1}(q)$ to $F^{-1}(\Psi_t(q))$, and its inverse Φ_{-t} similarly sends $F^{-1}(\Psi_t(q))$ to $F^{-1}(q)$. The restriction of Φ_t therefore gives us a diffeomorphism $F^{-1}(\Psi_t(q)) \approx F^{-1}(q)$. But since Ψ_t is just the clockwise rotation of S^1 through t radians, any $\tilde{q} \in S^1$ can be written as $\Psi_t(q)$ for some t, and we therefore have $F^{-1}(\tilde{q}) \approx F^{-1}(q)$ for any $q, \tilde{q} \in S^1$, as claimed.

6. Prove that there exists a smooth submersion $F : S^3 \to S^2$.

By identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, we can realize the 3-sphere as

$$S^3 = \{(z, \zeta) \in \mathbb{C}^2 \mid |z|^2 + |\zeta|^2 = 1\}.$$

This allows us to define a smooth map $F : S^3 \to \mathbb{CP}_1$ by

$$F(z, \zeta) = [z : \zeta].$$

This map is a submersion, because it has local smooth sections

$$[1:u] \to (\frac{e^{i\theta}}{\sqrt{|u|^2+1}}, \frac{e^{i\theta}u}{\sqrt{|u|^2+1}}), \quad \text{or} \quad [v:1] \mapsto (\frac{e^{i\theta}v}{\sqrt{|v|^2+1}}, \frac{e^{i\theta}}{\sqrt{|v|^2+1}})$$

passing through any given point of S^3. Since $\mathbb{CP}_1 \approx S^2$, the claim follows.

7. Let $p, q \in S^n \subset \mathbb{R}^{n+1}$ be the north and south poles $(0, \ldots, 0, \pm 1)$, and let $\Phi_1 : (S^n - \{p\}) \to \mathbb{R}^n$ and $\Phi_2 : (S^n - \{q\})) \to \mathbb{R}^n$ be the corresponding stereographic projections. Let $F : (\mathbb{R}^n - \{0\}) \to (\mathbb{R}^n - \{0\})$ be given by $F = \Phi_2 \circ \Phi_1^{-1}$. Compute the push-forward vector field $F_*(\partial/\partial x^1)$. Then use your computation to show that S^n carries a smooth vector field which only vanishes at one point.

The map $F : (\mathbb{R}^n - \{0\}) \to (\mathbb{R}^n - \{0\})$ is explicitly given by $F(\vec{x}) = \vec{y}$, where

$$y^j = \frac{x^j}{(x^1)^2 + \cdots + (x^n)^2}, \quad j = 1, \ldots, n.$$

The chain rule therefore tells us that

$$\begin{aligned}
F_*(\frac{\partial}{\partial x^1}) &= \sum_{j=1}^n \frac{\partial y^j}{\partial x^1} \frac{\partial}{\partial y^j} \\
&= \sum_{j=1}^n \frac{\partial}{\partial x^1} \left[\frac{x^j}{(x^1)^2 + \cdots + (x^n)^2} \right] \frac{\partial}{\partial y^j} \\
&= \sum_{j=1}^n \left(\frac{\delta_1^j}{(x^1)^2 + \cdots + (x^n)^2} - \frac{2x^1 x^j}{[(x^1)^2 + \cdots + (x^n)^2]^2} \right) \frac{\partial}{\partial y^j} \\
&= \sum_{j=1}^n \left([(y^1)^2 + \cdots + (y^n)^2]\delta_1^j - 2y^1 y^j \right) \frac{\partial}{\partial y^j} \\
&= \left[-(y^1)^2 + (y^2)^2 \cdots + (y^n)^2 \right] \frac{\partial}{\partial y^1} - 2y^1 \sum_{j=2}^n y^j \frac{\partial}{\partial y^j} .
\end{aligned}$$

This vector field extends smoothly across the origin, with value zero there. It follows that there is a smooth vector field on S^n which vanishes at exactly one point.