Problem 1. Submersions are open maps, hence, F is open. Now let C ≤ M be a closed set. Since M is compact, C is also compact. So F(C) is compact. So F(C) is closed. Hence, F is an open map, and F is also a closed map. Therefore, F(M) ⊆ N the image of F is both open and closed. Since M is non-empty, F(M) is non-empty, and since N is connected, F(M) = N, so F is onto. Since M is compact, so is F(M), so N is compact.

Problem 2. Consider the diffeomorphism F: \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
F(u, v) = (2u, v).
\]

And consider \( \tilde{g}: [0, \pi] \rightarrow \mathbb{R}^2 \)

\[
\tilde{g}(t) = (\cos(2\pi t), \sin(2\pi t)).
\]

Then \( (F \circ \tilde{g}) = \delta \), and

\[
F^* \phi = \frac{v d(2u) - 2uv dv}{4u^2 + 4v^2} = \frac{1}{2} \frac{vdu - udv}{u^2 + v^2}
\]

Now let us switch to polar coordinates in the \((u, v)\)-plane,

\[
\begin{align*}
\begin{cases}
u = r \cos \theta \\ v = r \sin \theta
\end{cases} 
& \\
\begin{cases}
u^2 + v^2 = r^2 \\ 0 < \theta < \pi
\end{cases}
& \\
F^* \phi = \frac{1}{2} \frac{vdu - udv}{u^2 + v^2} = \frac{1}{2} \frac{1}{r^2} \left( -r^2 \sin^2 \theta d\theta \\ + r^2 \theta \cos \theta dr - r \cos \theta \sin \theta dr - r^2 \cos^2 \theta d\theta \right)
\end{align*}
\]

\[
= - \frac{1}{2} d\theta
\]
Now, \( \int \varphi = \int \varphi = \int F^* \varphi \) and it is now clear that the integral is \( -\frac{1}{2} (2\pi) = -\pi \). But to be perfectly rigorous, we must introduce another chart \( \tilde{U}. \tilde{V} = \mathbb{R}^2 \setminus \text{origin, } \) and coordinates \( x = \tilde{r} \cos \tilde{\theta} \), \( y = \tilde{r} \sin \tilde{\theta} \) \( 0 < \tilde{\theta} < 2\pi \).

Clearly, on \( U \cap \tilde{U} = \mathbb{R}^2 \setminus \{x-axis\} \), \( \tilde{r} = \tilde{r} \), \( \tilde{\theta} = \tilde{\theta} \) and so \( d\theta = d\tilde{\theta}. \) Now, \( \int F^* \varphi = \int F^* \varphi + \int F^* \varphi \)

where \( \tilde{\sigma}_1 : [0, \frac{1}{2}] \rightarrow \mathbb{R}^2 \) \( \tilde{\sigma}_1(t) = (\cos(2\pi (t-\frac{1}{4})), \sin(2\pi (t-\frac{1}{4})) \) and \( \tilde{\sigma}_2 : [\frac{1}{2}, 1] \rightarrow \mathbb{R}^2 \) \( \tilde{\sigma}_2(t) = (\cos(2\pi (t-\frac{1}{4})), \sin(2\pi (t-\frac{1}{4})) \) and the image of \( \tilde{\sigma}_1 \subset U \), and the image of \( \tilde{\sigma}_2 \subset \tilde{U} \), so

\[
\int \tilde{\sigma}_1 \varphi = \int \tilde{\sigma}_1 F^* \varphi + \int \tilde{\sigma}_2 F^* \varphi = -\frac{1}{2} \left( \int \varphi d\theta + \int \varphi d\tilde{\theta} \right) = -\frac{1}{2} \left( \Theta(\tilde{\sigma}_1(1)) - \Theta(\tilde{\sigma}_1(0)) + \tilde{\sigma}_2(1) - \tilde{\sigma}_2(0) \right) = -\frac{1}{2} \left( \frac{\pi}{2} - \frac{\pi}{2} + \frac{3\pi}{2} - \frac{\pi}{2} \right) = -\frac{1}{2} (2\pi) = -\pi
\]

Further, \( \varphi \) is closed since \( F^* d\varphi \big|_U = d(F^* \varphi) \big|_U = d(-\frac{1}{2} d\varphi) \big|_U = 0 \) and similarly on \( \tilde{U} \). But \( F^* \) is an isomorphism, since \( F \) is a diffeomorphism, so \( F^* d\varphi = 0 \implies d\varphi = 0 \). However, \( \varphi \) is not exact, since if \( \varphi \) were exact then, there would be a function \( f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R} \) s.t. \( \varphi = df \). But then \( \int \varphi = \int df = f(\tilde{\sigma}(1)) - f(\tilde{\sigma}(0)) = 0 \neq -\pi \).
Problem 3. a) \( d(F^*\omega) = F^*(d\omega) = F^*(0) = 0 \)

The first equality follows because \( d \) commutes with pullback under a smooth map, the second equality follows since \( \omega \in \mathcal{A}^n(N) \) and \( \dim N = n \), so \( \mathcal{A}^{n+1}(N) = \{0\} \)
so \( d\omega \in \mathcal{A}^{n+1}(N) \) is automatically 0.

b. The point of the problem is that if we let

\[ F: \mathbb{R}^3 \setminus \{(0,0,0)\} \rightarrow S^2 \]
be given by \( F(x,y,z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} \)
and let \( \omega \) be the area form on \( S^2 \) induced from \( \mathbb{R}^3 \), then

\[ F^*\omega = \psi. \]

Let's expand on this and make the argument rigorous. Let \( f: \mathbb{R}^3 \to \mathbb{R} \) be given by \( f(x,y,z) = x^2 + y^2 + z^2 - 1 \).
Then \( df = 2xdx + 2ydy + 2zdz \), so \( df(x,y,z) = 0 \) only at the origin.

In particular, \( 0 \) is a regular value and so \( f^{-1}(0) = S^2 \subseteq \mathbb{R}^3 \)
is an embedded submanifold, and \( T_pS^2 \subseteq T_p\mathbb{R}^3 \) is equal to \( \ker(df|_p) \).
Define \( \omega(X,Y) = \text{signed area of the parallelogram spanned by } X,Y \in T_pS^2 \subseteq T_p\mathbb{R}^3 \), or formally,
if \( X = x^1 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} + x^3 \frac{\partial}{\partial z} \) and \( Y = y^1 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + y^3 \frac{\partial}{\partial z} \) and \( p = (a^1,a^2,a^3) \), then \( df_p(X_p) = a^2 \delta_{i3} - a^3 \delta_{i2} = 0 \), \( df_p(Y_p) = a^1 \delta_{i3} - a^2 \delta_{i2} = 0 \),

\[ \omega_p(X_p,Y_p) = \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ a^1 & a^2 & a^3 \end{vmatrix} \]
(this is actually the volume of the parallelopiped with base the parallelogram spanned by \( X,Y \) and height the unit vector \( (a^1,a^2,a^3) \) which is orthogonal to \( X,Y \)).

\( \omega \) so defined is clearly a 2-form on \( T_pS^2 \) and is smooth in \( p \).
Now we need to check that \( F^*\omega = \psi \). We can do this directly, but it is quicker to use a trick.
First, note that \( \psi = \frac{1}{p^3} \left( xdy \wedge dz + ydz \wedge dx + zdx \wedge dty \right) \in \Omega^2(R^3) \)

is rotationally invariant (we used \( p = (x^2 + y^2 + z^2)^{1/2} \)), i.e., for any \( T \in SO(3) \), \( T^* \psi = \psi \). Indeed, for a rotation \( T_{z, \theta} \) by angle \( \theta \) around the z-axis \( T_{z, \theta}^* \psi = T_{z, \theta}^* \frac{1}{p^3} \left( xdy \wedge ydx + zdz \wedge dx + zdx \wedge dty \right) \)

\( = \frac{1}{p^3} T_{z, \theta}^* (r^2 \theta \wedge dz + zdx \wedge dty) = \frac{1}{p^3} (r^2 \theta \wedge dz + zdx \wedge dty) \),

where we used \( r = (x^2 + y^2)^{1/2} \), and the fact that \( p, r, z \) are fixed by \( T_{z, \theta} \), and \( T_{z, \theta}^* d \theta = d \theta \) and \( T_{z, \theta}^* (dx \wedge dy) = dx \wedge dy \) since the latter is the area form on \( R^2 \), which is preserved by a rotation around the z-axis. Since \( \psi \) is unchanged by a cyclic permutation of the axes, \( \psi \) is invariant under rotations around the x and y axes as well, and hence, rotationally invariant as claimed.

Now, \( F \) commutes with rotations, and \( \omega \) is obviously rotationally invariant, so we can check that \( F^* \left( 1, \frac{\partial}{\partial x} \right) \omega \left( 1, \frac{\partial}{\partial x} \right) = \frac{1}{t^2} dy \wedge dx \)

to show that \( F^* \omega = \psi \) everywhere. Compute:

\( \left( \frac{1}{t^2} dy \wedge dx \right)(X, Y) = \frac{1}{t^2} (x^2 y^3 - x^3 y^2) \)

\( F^* \left( 1, \frac{\partial}{\partial x} \right) \omega \left( 1, \frac{\partial}{\partial x} \right) (X, Y) = \omega \left( 1, \frac{\partial}{\partial x} \right) (F^* \left( 1, \frac{\partial}{\partial x} \right) X, F^* \left( 1, \frac{\partial}{\partial x} \right) Y) = \omega \left( 1, \frac{\partial}{\partial x} \right) \left( F_{(1, x)} X, F_{(1, x)} Y \right) = \omega \left( 1, \frac{\partial}{\partial x} \right) \left( x^2 \frac{\partial}{\partial y} + x^3 \frac{\partial}{\partial z}, y^2 \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial z} \right) = \frac{1}{t^2} \left( \begin{array}{ccc} 0 & x^2 & x^3 \\ \frac{\partial}{\partial x} & y^2 & y^3 \\ x^2 \frac{\partial}{\partial y} + x^3 \frac{\partial}{\partial z} & y^2 \frac{\partial}{\partial x} + y^3 \frac{\partial}{\partial z} \end{array} \right) = \frac{1}{t^2} \left( x^2 y^3 - x^3 y^2 \right). \)

Thus, since \( \psi = F^* \omega \), \( \omega \in \Omega^2(S^2) \), by part (a) \( d \psi = d(F^* \omega) = F^* (d \omega) = F^* (0) = 0 \).
Problem 4.a. Suppose $\beta$ is exact so $\exists f \in C^\infty(M)$ s.t. $\beta = df$.

Since $M$ is compact, $f$ attains its maximum on $M$, at say $p \in M$. By elementary calculus $\forall X \in T_p M \quad X(f)_p = 0$

But $0 = X(f)|_p = df|_p (x) = \beta_p (x)$, so $\beta_p = 0$, contradicting the non-vanishing of $\beta$.

b. As we know in Problem 2, $\exists \theta \in A^1(S^1)$ a non-vanishing 1-form on $S^1$. (Note, there is no $\theta \in C^\infty(S^1)$ s.t. the 1-form in question is actually $d\Theta$, but $\Theta = \text{Tan}^{-1}(x) \in C^\infty(\mathbb{R})$,

$\mathcal{U} = \{(x,y) \in S^1 \mid x \neq 0\}$, and $\tilde{\Theta} = \frac{1}{2} \text{Tan}^{-1}(xy) \in C^\infty(\mathbb{R})$,

$\tilde{\mathcal{U}} = \{(x,y) \in S^1 \mid y \neq 0\}$, and $\mathcal{U} \cap \tilde{\mathcal{U}} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$

and $\Theta - \tilde{\Theta} = C_1$, where $C_1$ is a constant, and $Q_i$ are the 4 quadrants excluding the axes).

Let $\beta \in A^1(M)$ be given by $F^* d\Theta$, then $\beta$ is non-vanishing.

Indeed, if $p \in M$ and $\beta_p = 0$, then $\forall X \in T_p M \quad \beta_p (X) = 0$, but

$0 = \beta_p (X) = F_p^* (d\Theta |_{F_p(S^1)}) (X) = d\Theta |_{F_p(S^1)} (F_p X)$, but $F$ is a submersion, so $F_p$ is onto $T_{F_p(S^1)}$, so $\exists X \in T_p M$ s.t. $F_p X = \frac{\partial}{\partial y} |_{F_p(S^1)}$

and so $d\Theta |_{F_p(S^1)} (F_p X) = d\Theta |_{F_p(S^1)} (\frac{\partial}{\partial y} |_{F_p(S^1)}) = 0$, contradicting.

But if $\beta$ is non-vanishing, by part a, it is not exact.

On the other hand by 3a, $\beta$ is closed ($d\beta = d(F^* d\Theta) = F^* d(d\Theta) = 0$ since $d(d\Theta) \in A^2(S^1) = \{0\}$). Hence, $[\beta]$ is a nonzero element of the space of closed 1-forms modulo the subspace of exact 1-forms, so $[\beta] \neq 0$ in $H^1_{deRham}(M)$. 

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Problems 5. Let \( g \in \mathbb{C} \). Then \( g \) is continuous

We fix a coordinate neighborhood \( (x', \ldots, x^n) : U \to \mathbb{R}^n \)

R.e.: \((x', \ldots, x^n)\) are continuous on \( M \cap U \) \( (n = \dim N, \)

\( m = \dim M \) and \( l = \dim L) \). Also \( \bar{U} \subset N \), a coordinate

neighborhood of \( g \) and \((x', \ldots, x^n) : \bar{U} \to \mathbb{R}^n \)

\( + (\tilde{x}', \ldots, \tilde{x}^n) \) are continuous on \( L \cap \bar{U} \). Let \( V = U \cap \bar{U} \)

and restrict the \( x' \) and \( x^n \) to \( V \). Then \( MNV = \)

\( \{ p \in V \mid x^{m+1}(p) = \ldots = x^n(p) = 0 \} \) and \( L \cap V = \)

\( \{ p \in V \mid \tilde{x}^{l+1}(p) = \ldots = \tilde{x}^n(p) = 0 \} \). Now consider

\( F : V \to \mathbb{R}^{2n-m-l}, \) \( F(p) = (x^{m+1}(p), \ldots, x^n(p), \tilde{x}^{l+1}(p), \ldots, \tilde{x}^n(p)) \).

Then \( LN M N V = F^{-1}(0, \ldots, 0) \). Claim: \((0, \ldots, 0) \)

is a regular value of \( F \). Indeed, \( \forall q \in LN M N V, \exists \Sigma \subset \Sigma \in T_q M \)

\( (DF_q) \xi = (dx^{m+1}_q(\xi), \ldots, dx^n_q(\xi), dx^{l+1}_q(\xi), \ldots, dx^n_q(\xi)) \)

If \( (DF_q) \xi = 0 \), then \( dx^i_q(\xi) = 0 \) for \( i \leq m \) and

\( dx^{l+1}_q(\xi) = 0 \) for \( l+1 \leq j \leq n \), so \( \xi \in T_q M \) and \( \xi \in T_q L \).

And obviously if \( \xi \in T_q M \cap T_q L \) then \( (DF_q) \xi = 0 \).

So \( \ker DF_q = T_q M \cap T_q L \). Now, \( T_q M + T_q L = T_q N \)

So \( m + l - \dim (T_q M \cap T_q L) = n \), so \( \dim \ker DF_q = \)

\( n + m + l - \dim T_q M \). \( DF_q \) is a map from an \( n \)-dimensional space to

\( \mathbb{R}^{(m-m)+(n-l)} \), and since \( T_q M + T_q L = T_q N \), \( m + l > n \). So \( \dim \kerDF_q \) is

onto. So \( (0, \ldots, 0) \in \mathbb{R}^{2n-m-l} \) is a regular value, so

\( F^{-1}(0, \ldots, 0) = LN M N V \) is a manifold of dimension \( m + l - n \). It follows that \( LN M \) is a submanifold.