Emergent Conformal Structure

Computational Conformal Geometry Conference
Stony Brook, April 2007

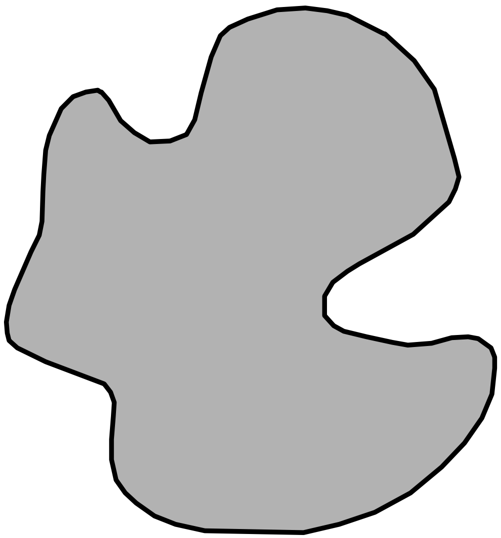
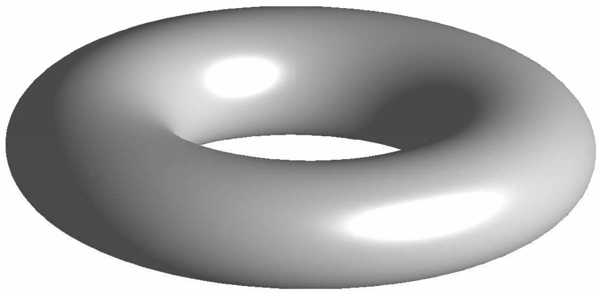
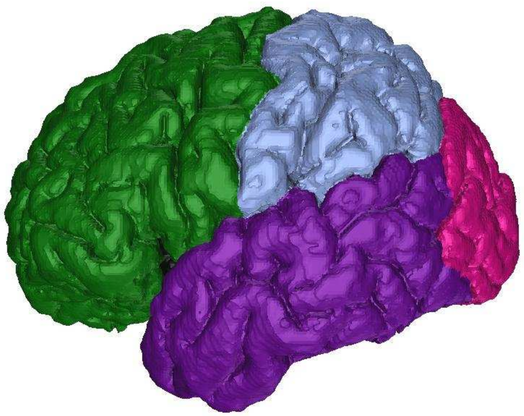
Ken Stephenson, University of Tennessee

Outline

- Background
- Discrete Conformal Geometry
- Emergent Conformal Geometry

Outline

- Background
- Discrete Conformal Geometry
- Emergent Conformal Geometry

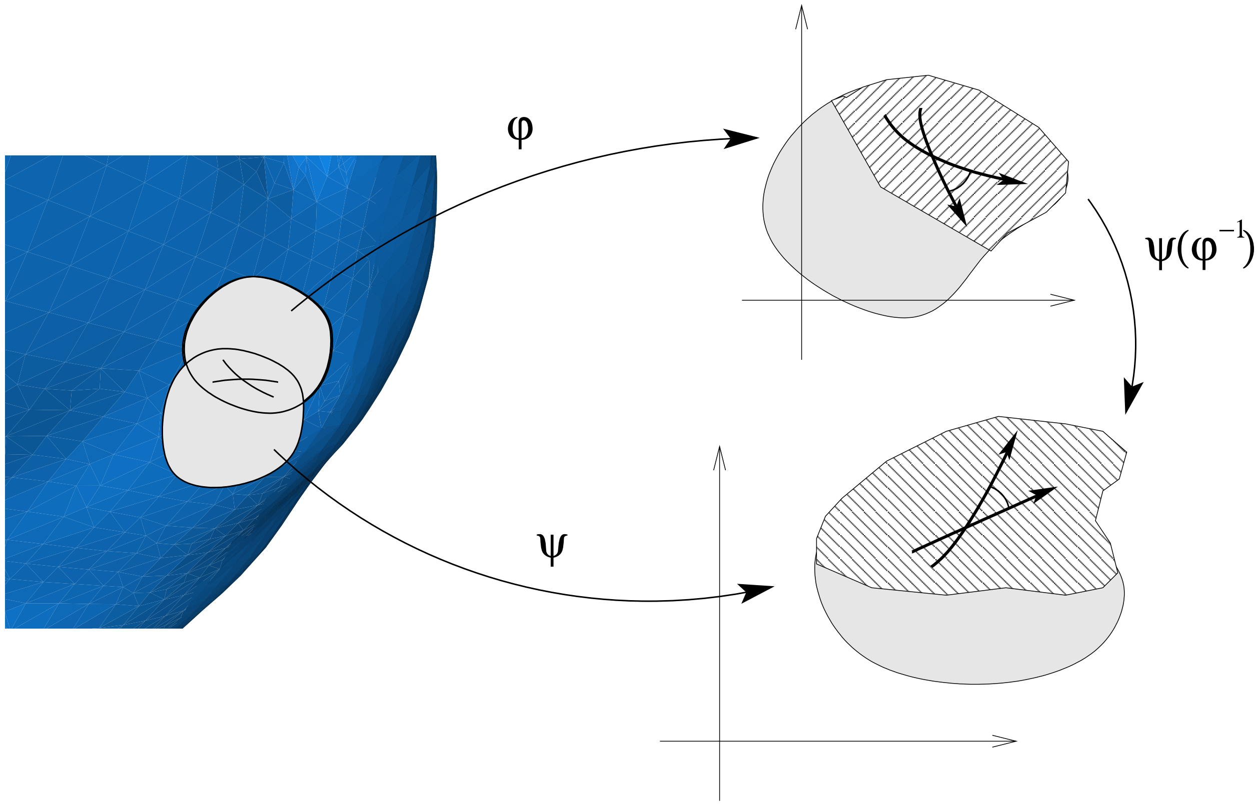




1. Background

- Conformal geometry
- Circle packing
- Enabling theory

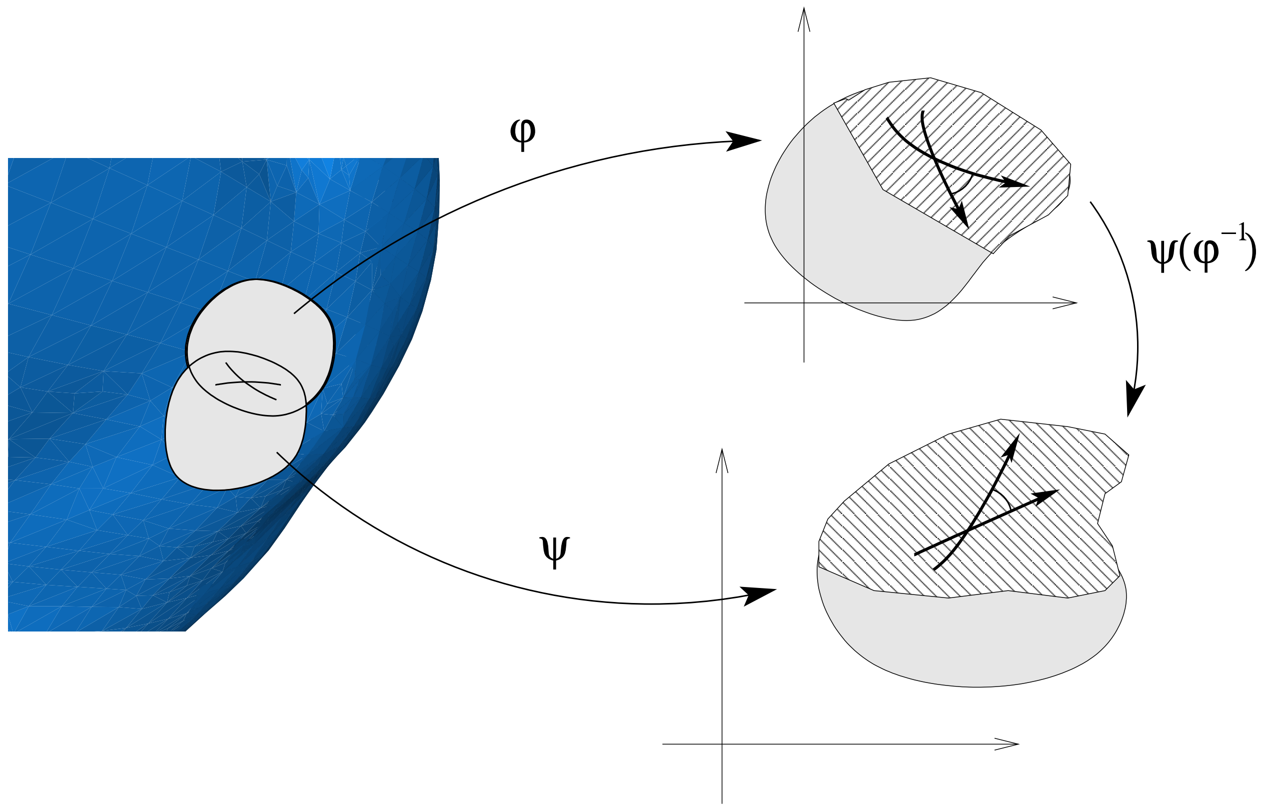
Classical Smoke-and-Mirrors

A Riemann surface is one with a consistent notion of “angle”.



Classical Smoke-and-Mirrors

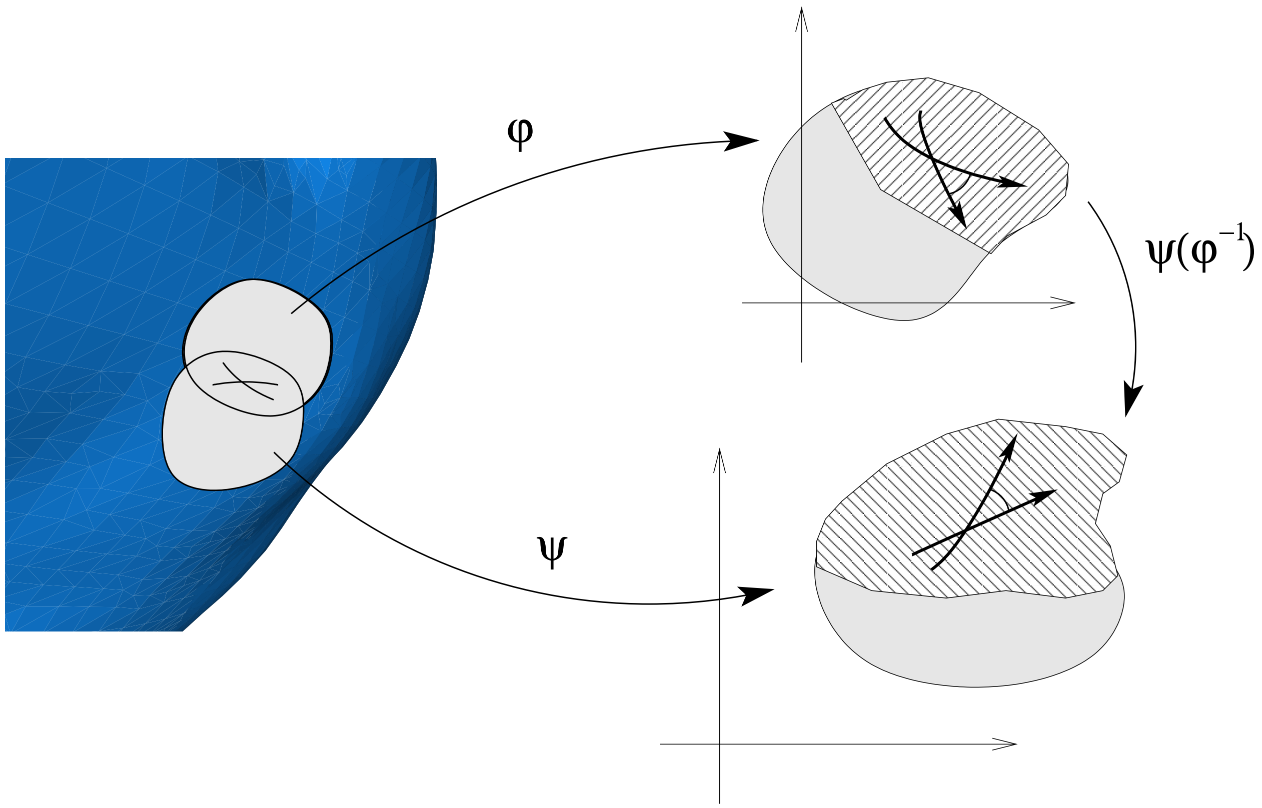
A Riemann surface is one with a consistent notion of “angle”.



“Conformal structure” refers to whatever resides in the web of consistency relationships defined by the conformal transition maps.

Classical Smoke-and-Mirrors

A Riemann surface is one with a consistent notion of “angle”.



“Conformal structure” refers to whatever resides in the web of consistency relationships defined by the conformal transition maps.

“Conformal maps” are maps between Riemann surfaces which preserve angles (magnitude and orientation).

Conformal Mapping is Ubiquitous

- **Riemann Mapping Theorem (1851):** *Every simply connected Riemann surface can be mapped conformally onto the sphere, the plane, or the unit disc, and the resulting map is unique up to Möbius transformations.*

Conformal Mapping is Ubiquitous

- **Riemann Mapping Theorem (1851):** *Every simply connected Riemann surface can be mapped conformally onto the sphere, the plane, or the unit disc, and the resulting map is unique up to Möbius transformations.*
- Extended via "covering theory" to handle Riemann surfaces in full generality.
- A core topic in mathematics
- Application in physics, engineering, visualization

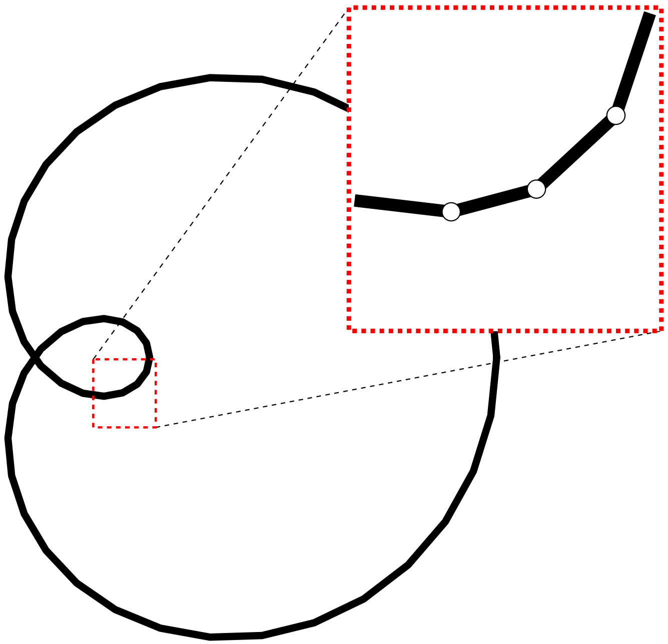
Conformal Mapping is Ubiquitous

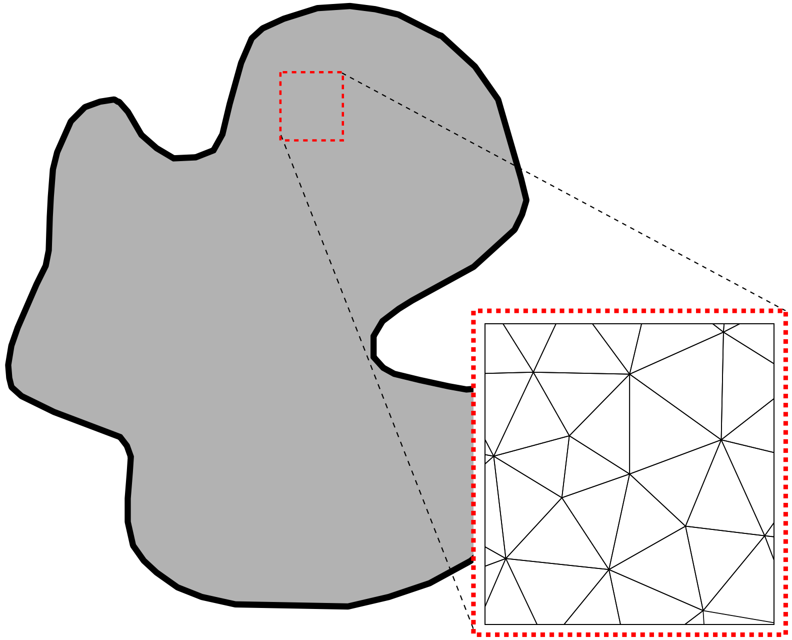
- **Riemann Mapping Theorem (1851):** *Every simply connected Riemann surface can be mapped conformally onto the sphere, the plane, or the unit disc, and the resulting map is unique up to Möbius transformations.*
- Extended via “covering theory” to handle Riemann surfaces in full generality.
- A core topic in mathematics
- Application in physics, engineering, visualization

PROBLEM? Practical computations.

Discretization

Discretization

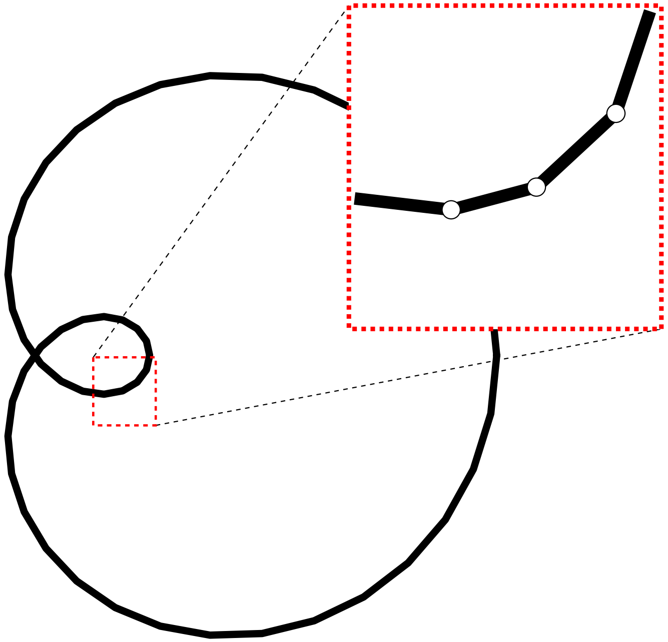


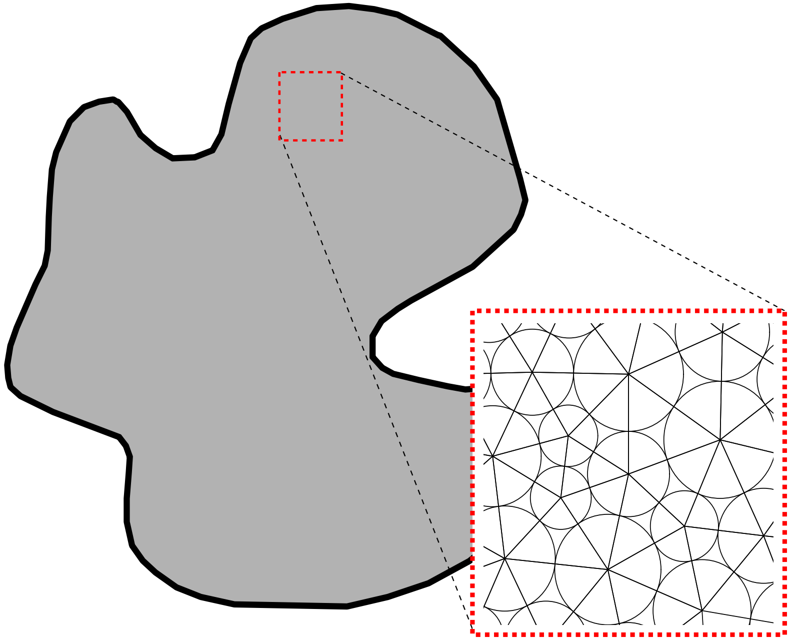


What we should hope for:

- Geometric intuition
- Discrete versions of classical objects
- Computability
- Refinement procedures
- Convergence to the classical objects

Discretization

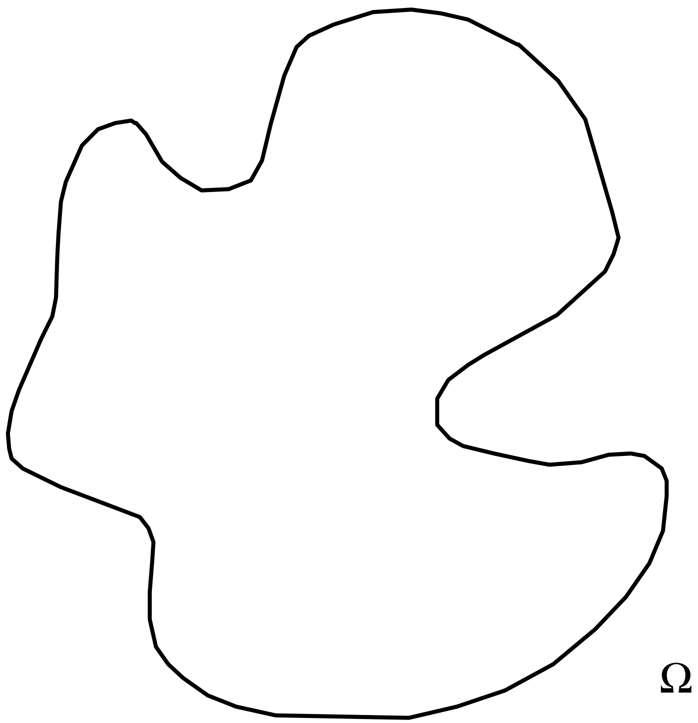




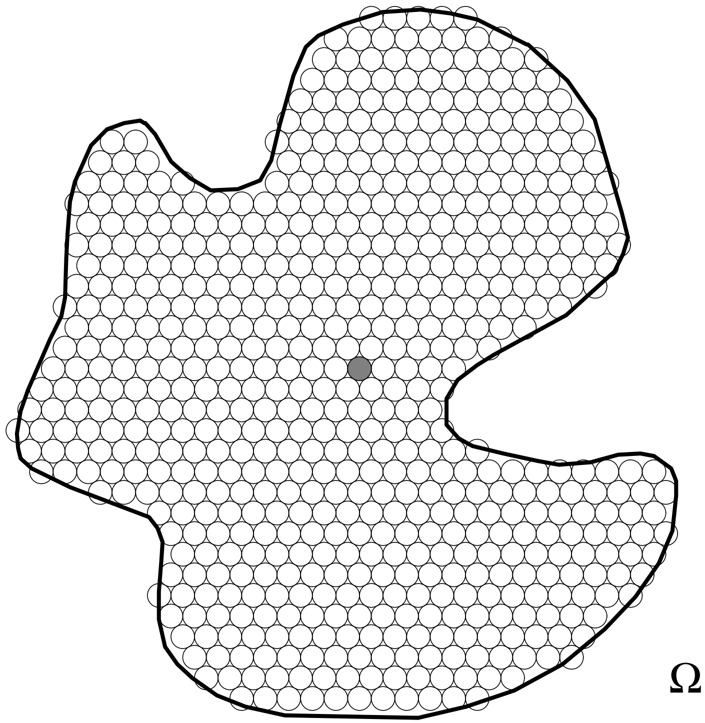
What we should hope for:

- Geometric intuition
- Discrete versions of classical objects
- Computability
- Refinement procedures
- Convergence to the classical objects

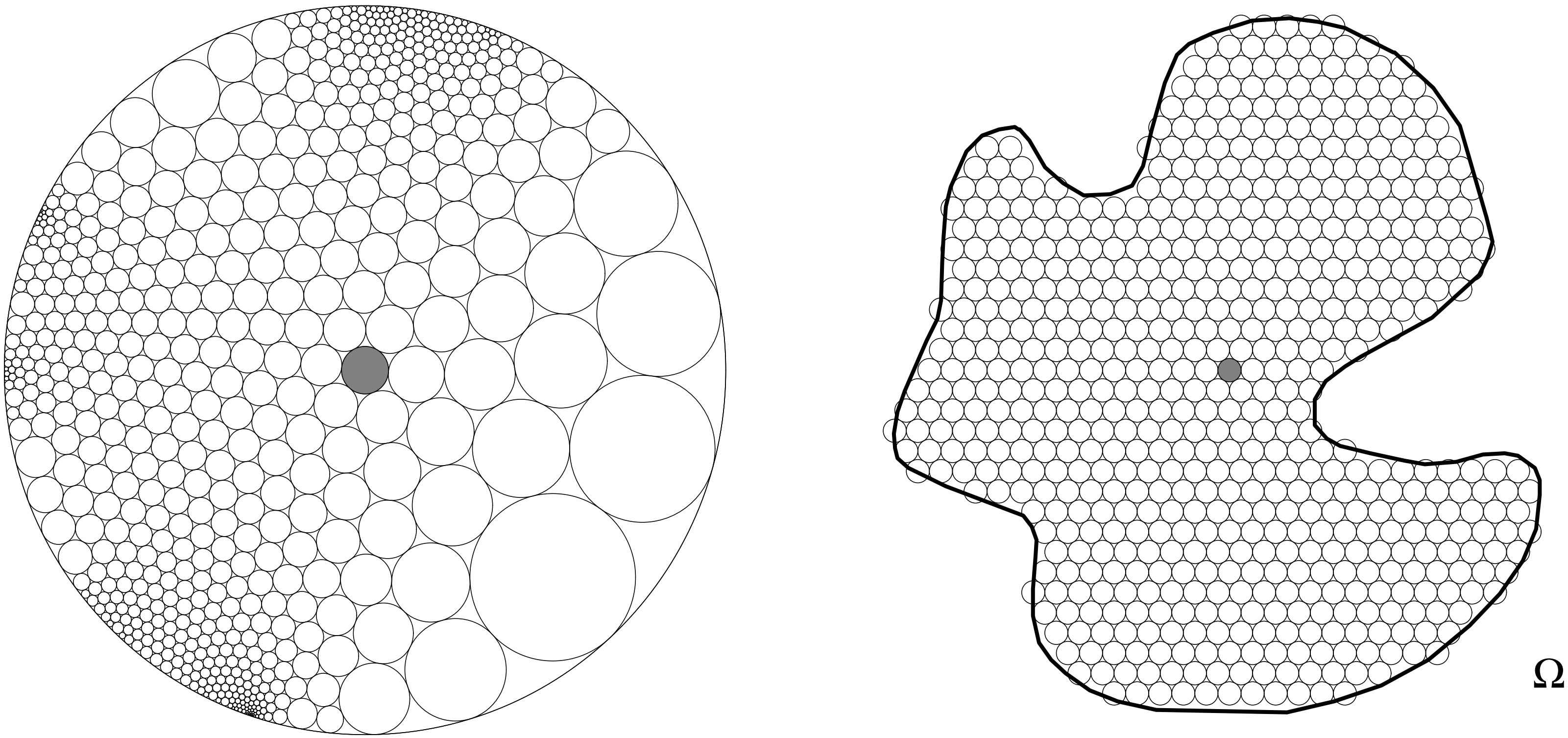
Thurston's Excellent Idea



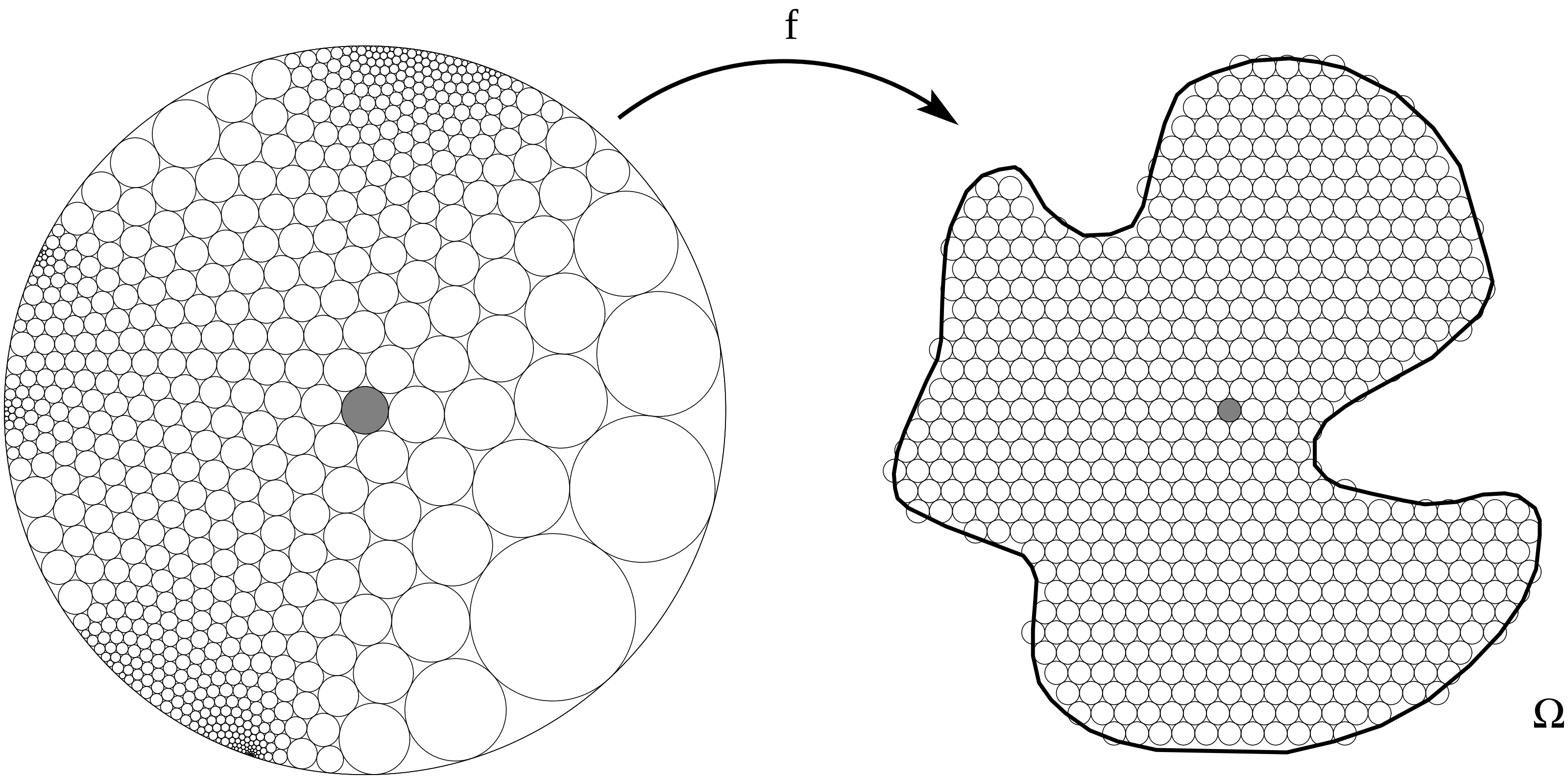
Thurston’s Excellent Idea



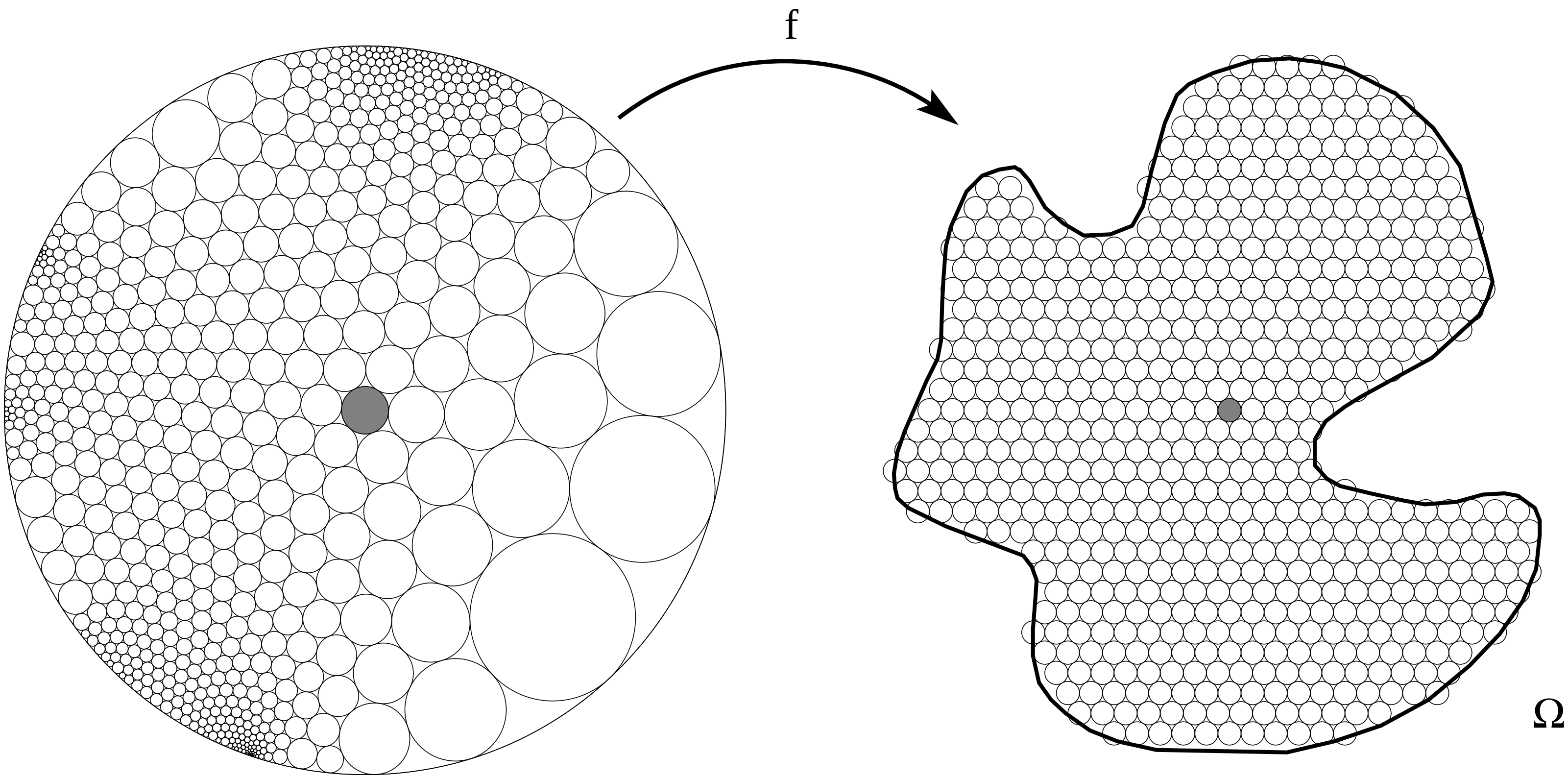
Thurston's Excellent Idea



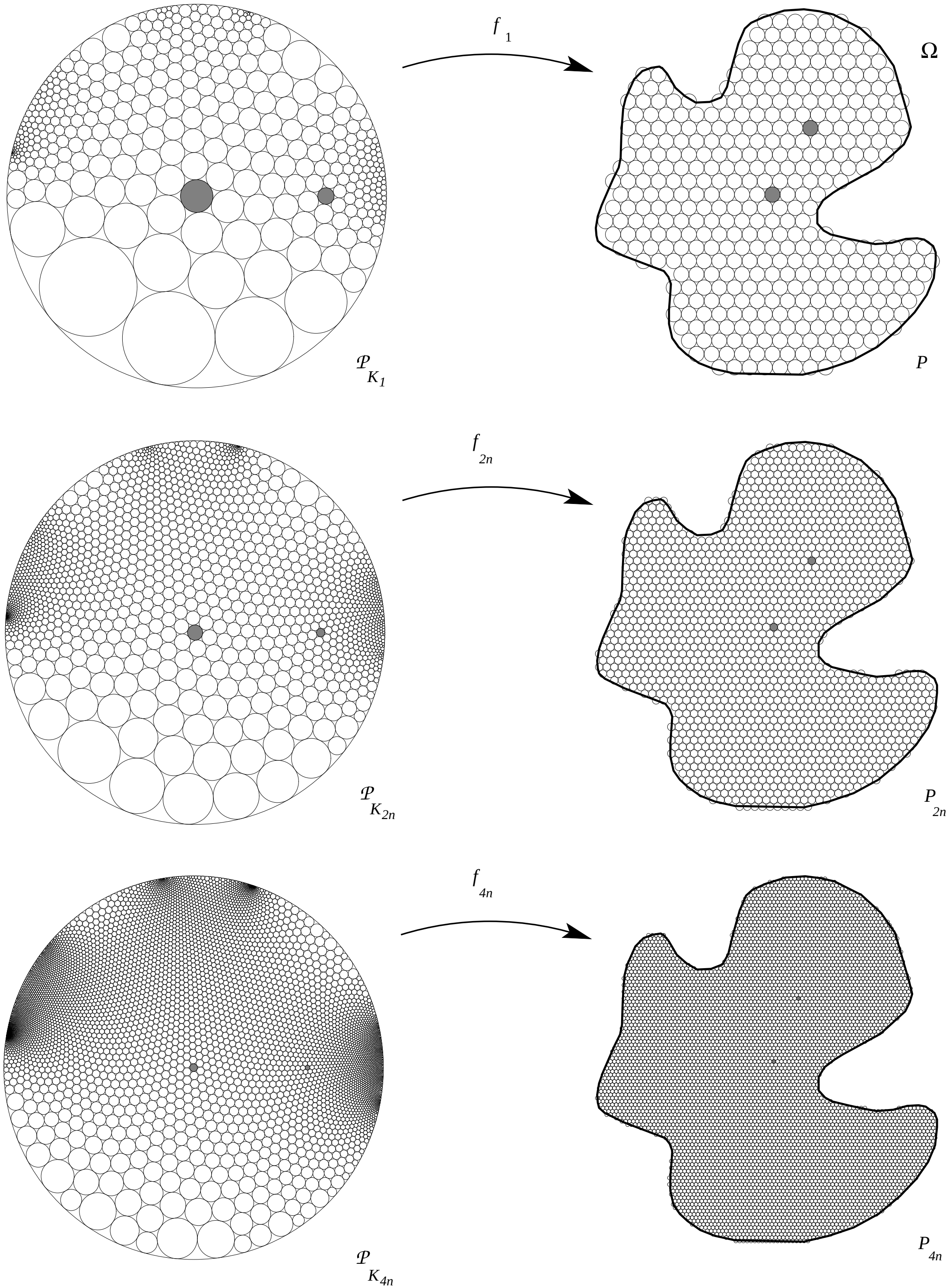
Thurston's Excellent Idea



Thurston's Excellent Idea



Thurston's Conjecture: *If increasingly fine hexagonal circle packings* P_n *are used in* Ω *and the maps* f_n *are appropriately normalized, then* f_n *converges uniformly on compact subsets of* $\mathbb{D}$ *to the classical conformal mapping* $F : \mathbb{D} \longrightarrow \Omega$*.*



f 1
Ω
P
f 2n
P 2n
f 4n
P 4n

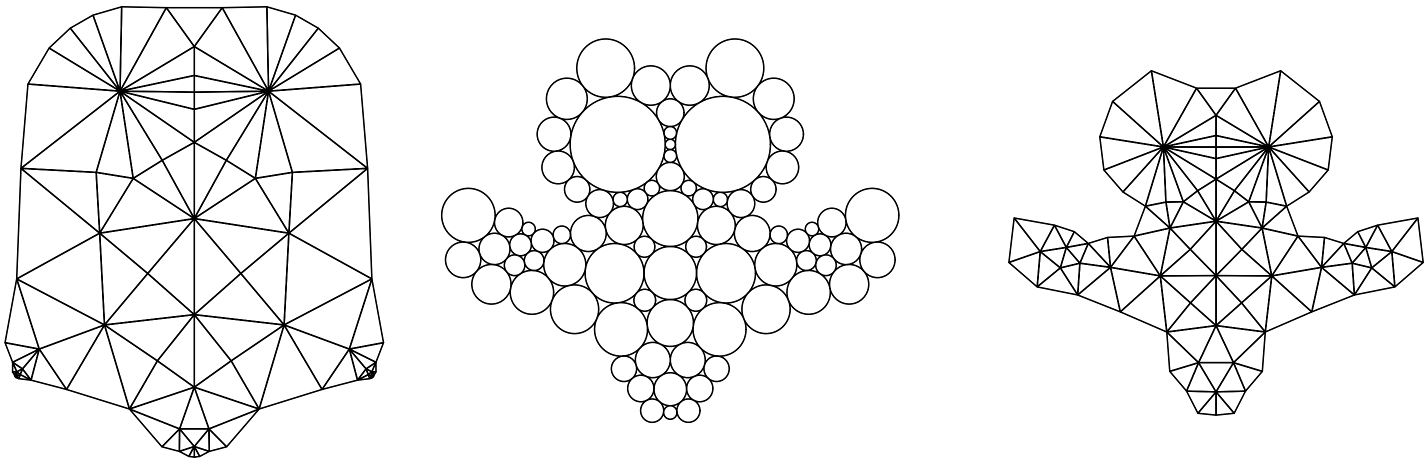
Circle Packing Basics

Def: A **circle packing** P is a configuration of circles with a specified pattern of tangencies. (Initiated by Koebe, Andreev, and (principally) Bill Thurston.)

Circle Packing Basics

Def: A **circle packing** P is a configuration of circles with a specified pattern of tangencies. (Initiated by Koebe, Andreev, and (principally) Bill Thurston.)

- The pattern of P is given by a (simplicial) **complex** K which triangulates an oriented topological surface.
- The configuration P has a circle C_v for each vertex $v \in K$. When $\langle u, v \rangle$ is an edge of K, then C_u and C_v are tangent. When $\langle u, v, w \rangle$ is an oriented face of K, then $\langle C_u, C_v, C_w \rangle$ is an oriented triple of mutually tangent circles.
- The radii are given in a **label** R. (Computing R is where the work goes; compatibility depends on **angle sums** — centers are secondary.)



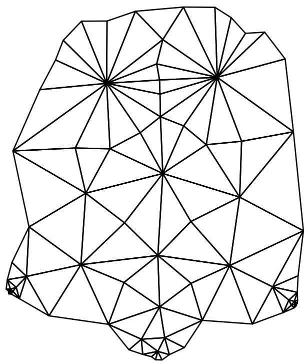
Typical operation: **given** $K \longrightarrow$ **compute** $R \longrightarrow$ **lay out** P

Packing Plasticity

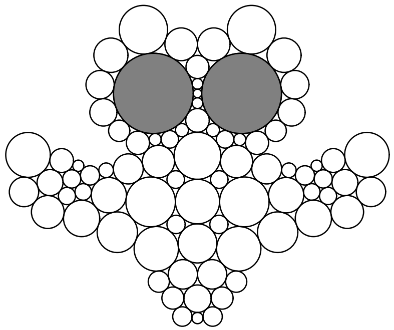
The theory has been extended with boundary conditions and branching (not pertinent here) to give amazing plasticity.

Packing Plasticity

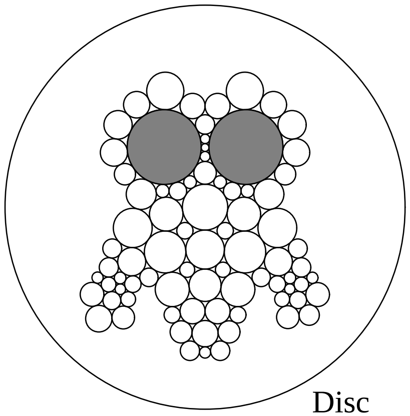
The theory has been extended with boundary conditions and branching (not pertinent here) to give amazing plasticity.



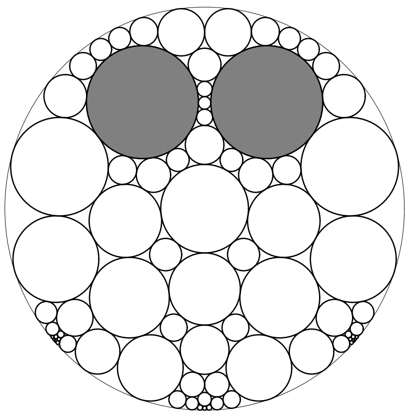
Common Combinatorics K



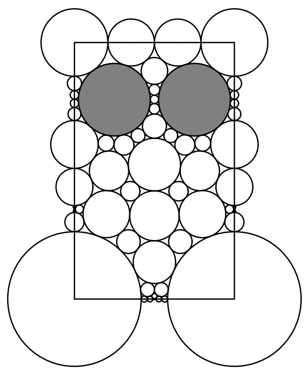
Specified boundary radii



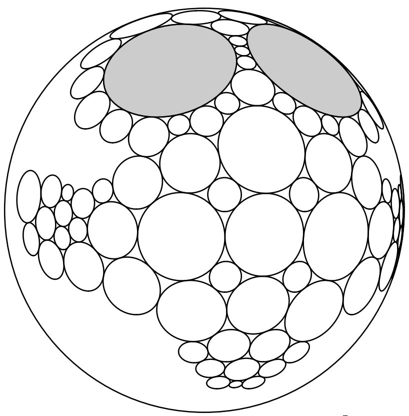
Disc



"Maximal" packing P_K



Specified Boundary angles



Sphere

Enabling Theory

- **Koebe-Andreev-Thurston Theorem:** For any triangulation K of a sphere, there exists an associated univalent circle packing $\widetilde{P}$ of the Riemann sphere, unique up to Möbius transformations

Enabling Theory

- **Koebe-Andreev-Thurston Theorem:** For any triangulation K of a sphere, there exists an associated univalent circle packing $\widetilde{P}$ of the Riemann sphere, unique up to Möbius transformations
- Thurston's Conjecture on convergence to conformal mapping: proved by Burt Rodin and Dennis Sullivan using quasiconformal mapping theory.

Enabling Theory

- **Koebe-Andreev-Thurston Theorem:** For any triangulation K of a sphere, there exists an associated univalent circle packing $\widetilde{P}$ of the Riemann sphere, unique up to Möbius transformations
- Thurston's Conjecture on convergence to conformal mapping: proved by Burt Rodin and Dennis Sullivan using quasiconformal mapping theory.
- Convergence extended by various authors to more general combinatorics, still using quasiconformal theory

Enabling Theory

- **Koebe-Andreev-Thurston Theorem:** For any triangulation K of a sphere, there exists an associated univalent circle packing $\widetilde{P}$ of the Riemann sphere, unique up to Möbius transformations
- Thurston's Conjecture on convergence to conformal mapping: proved by Burt Rodin and Dennis Sullivan using quasiconformal mapping theory.
- Convergence extended by various authors to more general combinatorics, still using quasiconformal theory
- Culminating in a theorem of Zheng-Xu He and Oded Schramm that removes the quasiconformal theory, implying:

Enabling Theory

- **Koebe-Andreev-Thurston Theorem:** For any triangulation K of a sphere, there exists an associated univalent circle packing $\widetilde{P}$ of the Riemann sphere, unique up to Möbius transformations
- Thurston's Conjecture on convergence to conformal mapping: proved by Burt Rodin and Dennis Sullivan using quasiconformal mapping theory.
- Convergence extended by various authors to more general combinatorics, still using quasiconformal theory
- Culminating in a theorem of Zheng-Xu He and Oded Schramm that removes the quasiconformal theory, implying:

> The Koebe-Andreev-Thurston Theorem is equivalent to the Riemann Mapping Theorem for plane domains.

And ...

And ...

- Circle packings, refinements, and convergence results are extended to Riemann surfaces

And ...

- Circle packings, refinements, and convergence results are extended to Riemann surfaces
- With notion of branch points, circle packings provide wide ranging “discrete analytic functions”: discrete rational maps, inner functions, entire functions, covering maps, etc.

And ...

- Circle packings, refinements, and convergence results are extended to Riemann surfaces
- With notion of branch points, circle packings provide wide ranging “discrete analytic functions”: discrete rational maps, inner functions, entire functions, covering maps, etc.
- Indeed, a fairly comprehensive theory of discrete analytic functions emerges:

And ...

- Circle packings, refinements, and convergence results are extended to Riemann surfaces
- With notion of branch points, circle packings provide wide ranging “discrete analytic functions”: discrete rational maps, inner functions, entire functions, covering maps, etc.
- Indeed, a fairly comprehensive theory of discrete analytic functions emerges:

Circle Packing: “quantum” complex analysis, classical in the limit.

And ...

- Circle packings, refinements, and convergence results are extended to Riemann surfaces
- With notion of branch points, circle packings provide wide ranging “discrete analytic functions”: discrete rational maps, inner functions, entire functions, covering maps, etc.
- Indeed, a fairly comprehensive theory of discrete analytic functions emerges:

Circle Packing: “quantum” complex analysis, classical in the limit.

Important to our story:

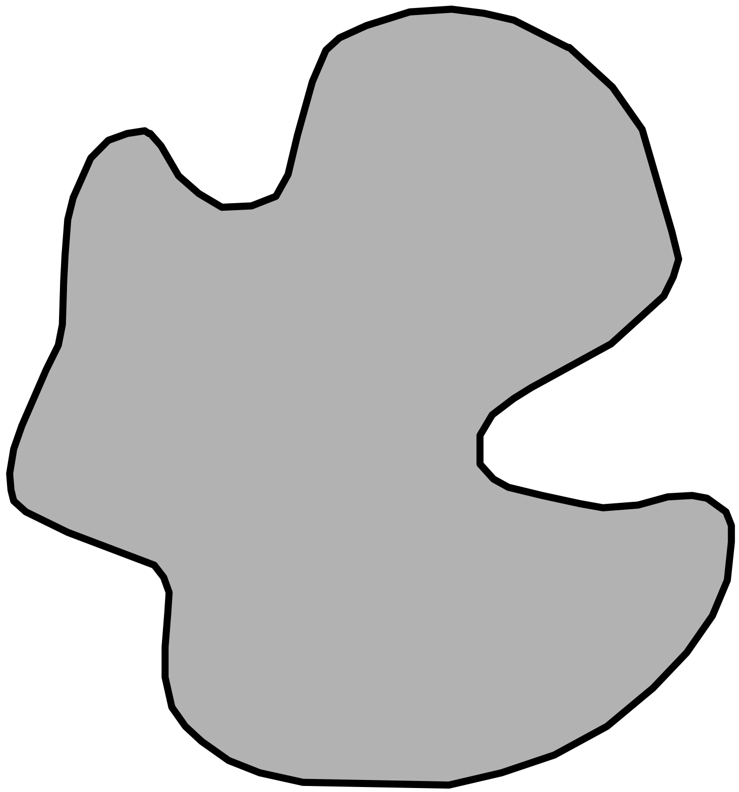
the existence of practical (and provable) algorithms for computing circle packings and software **CirclePack** for manipulating them.

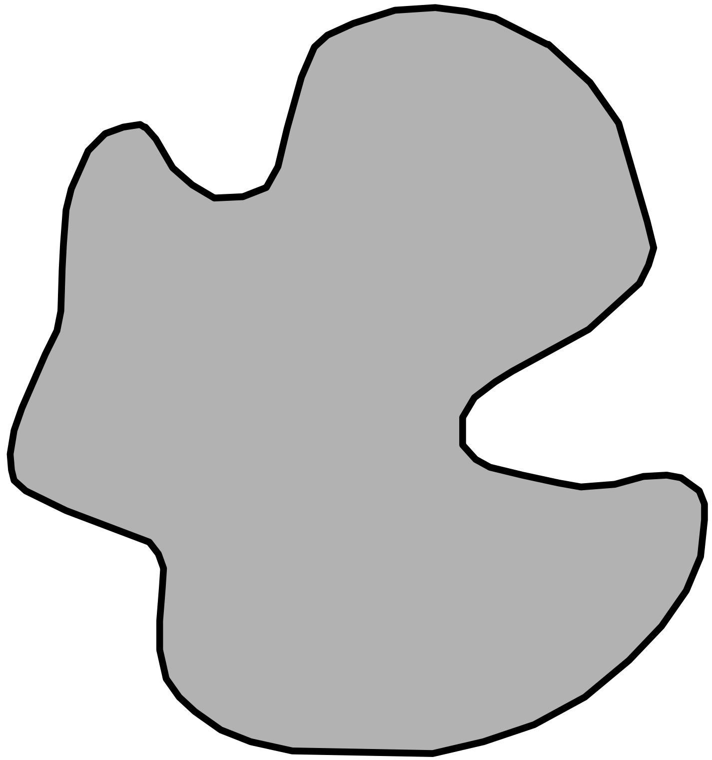
2. Discrete Conformal Structure

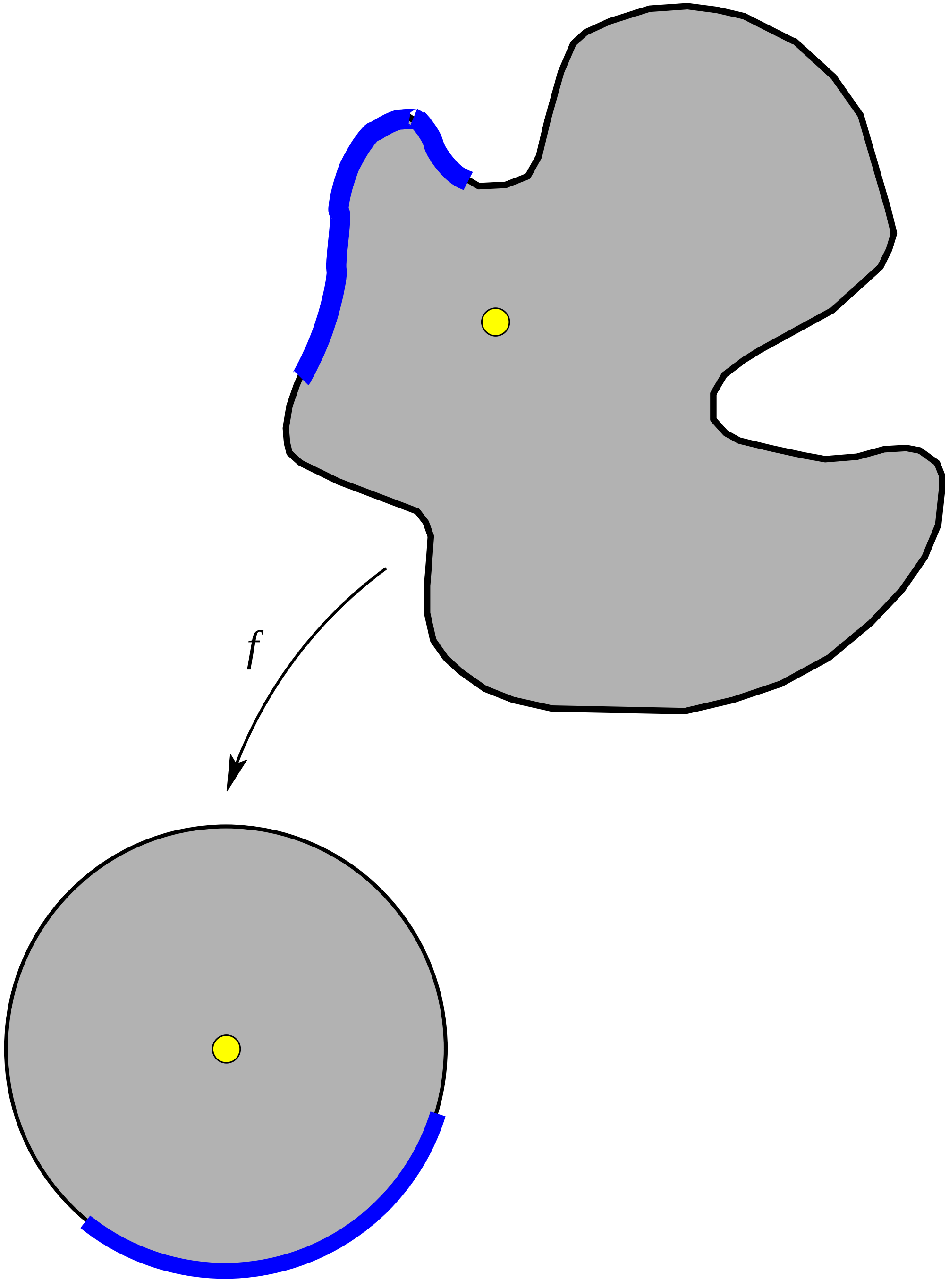
- Classical conformal companions
- Discrete versions
- Discrete conformal structure

Classical Conformal Structure — and Companions

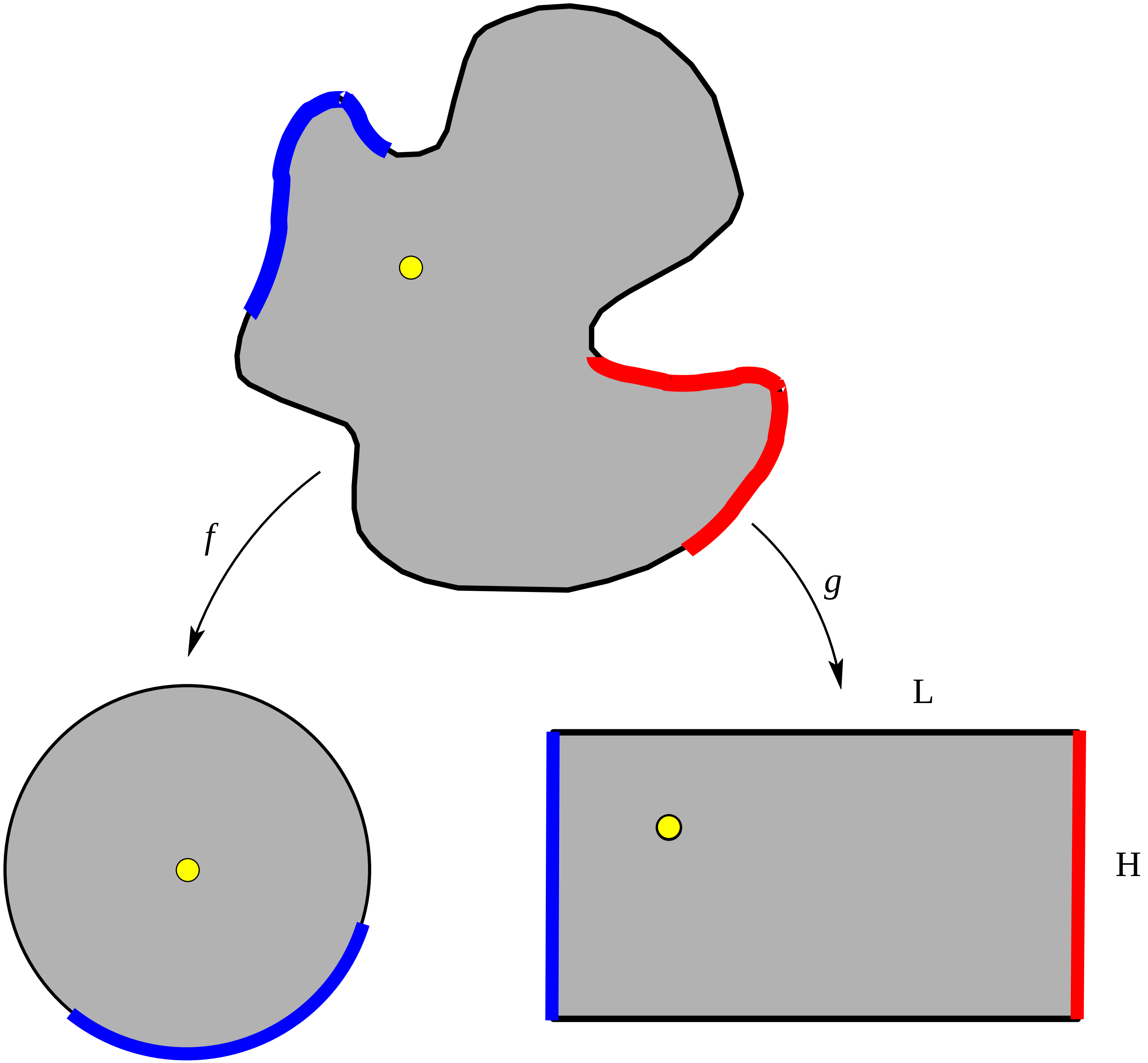
- Conformal maps
- Brownian motion
- Harmonic measure
- Extremal length







harmonic measure

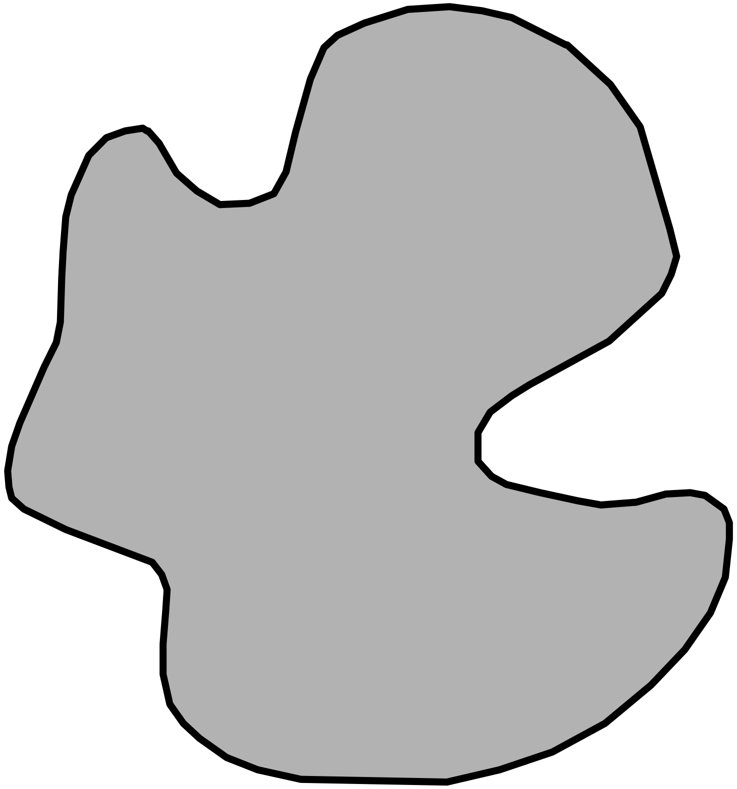


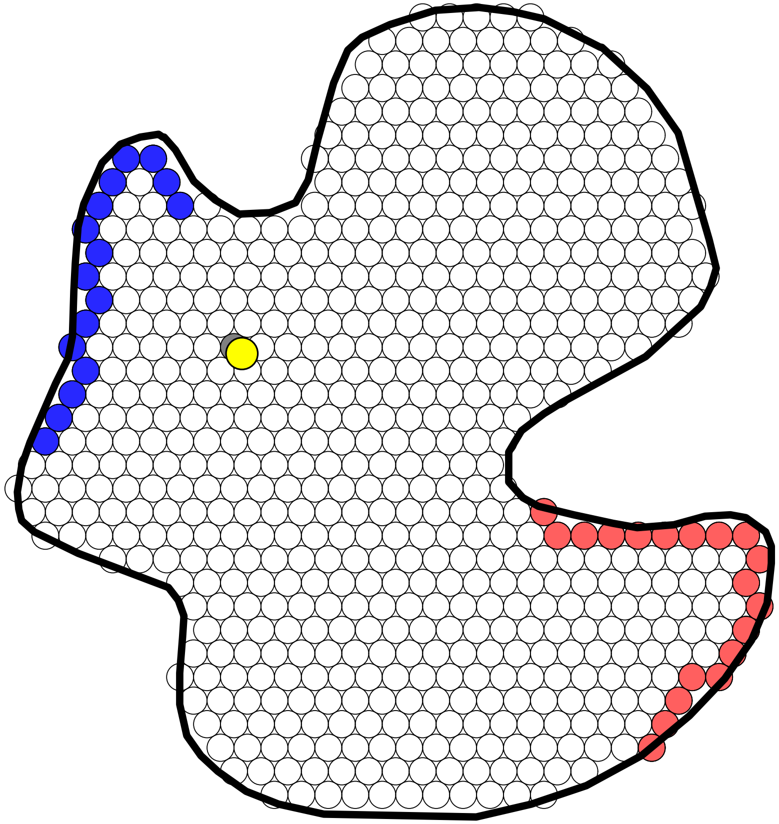
harmonic measure

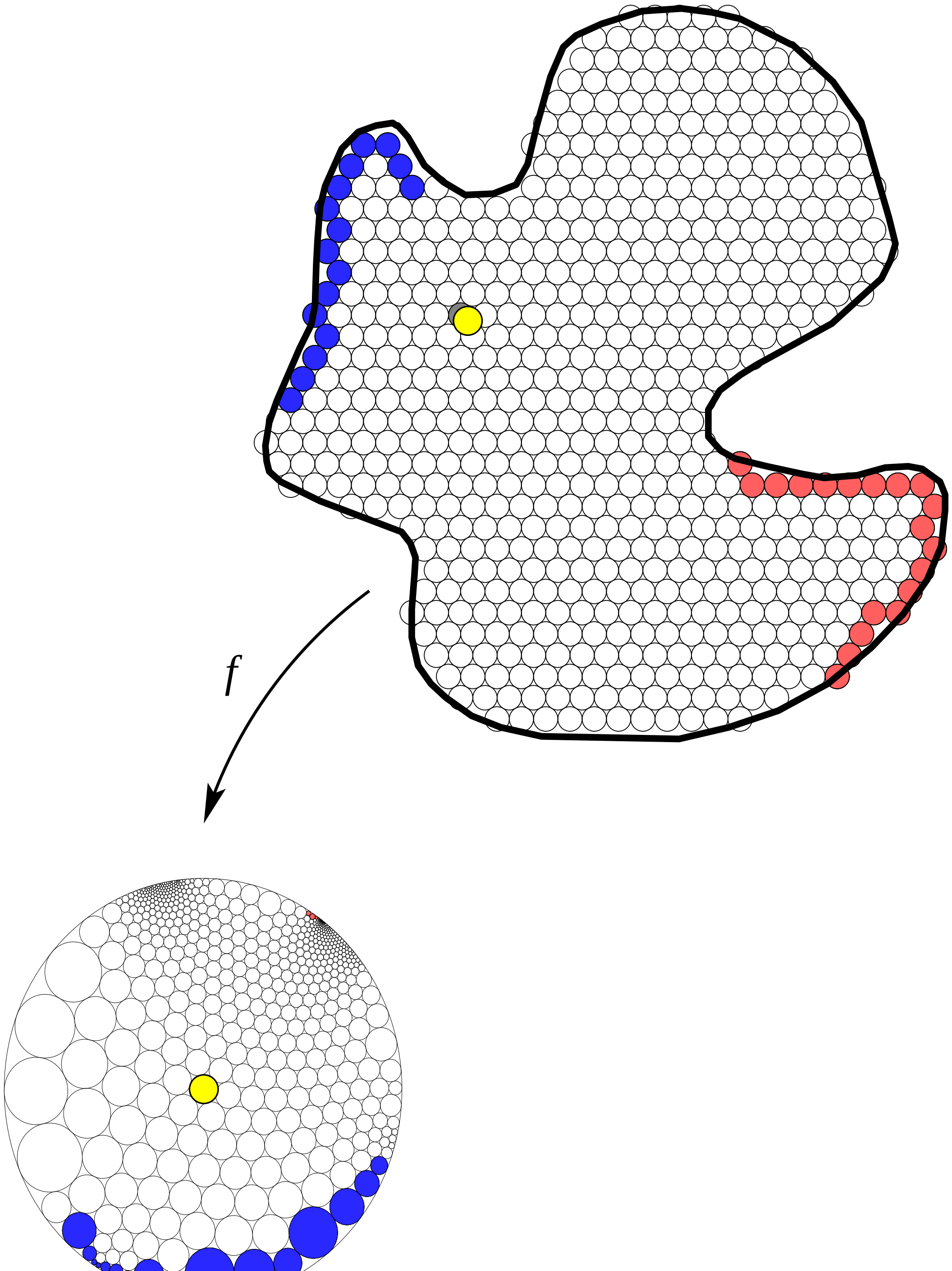
extremal length = L/H

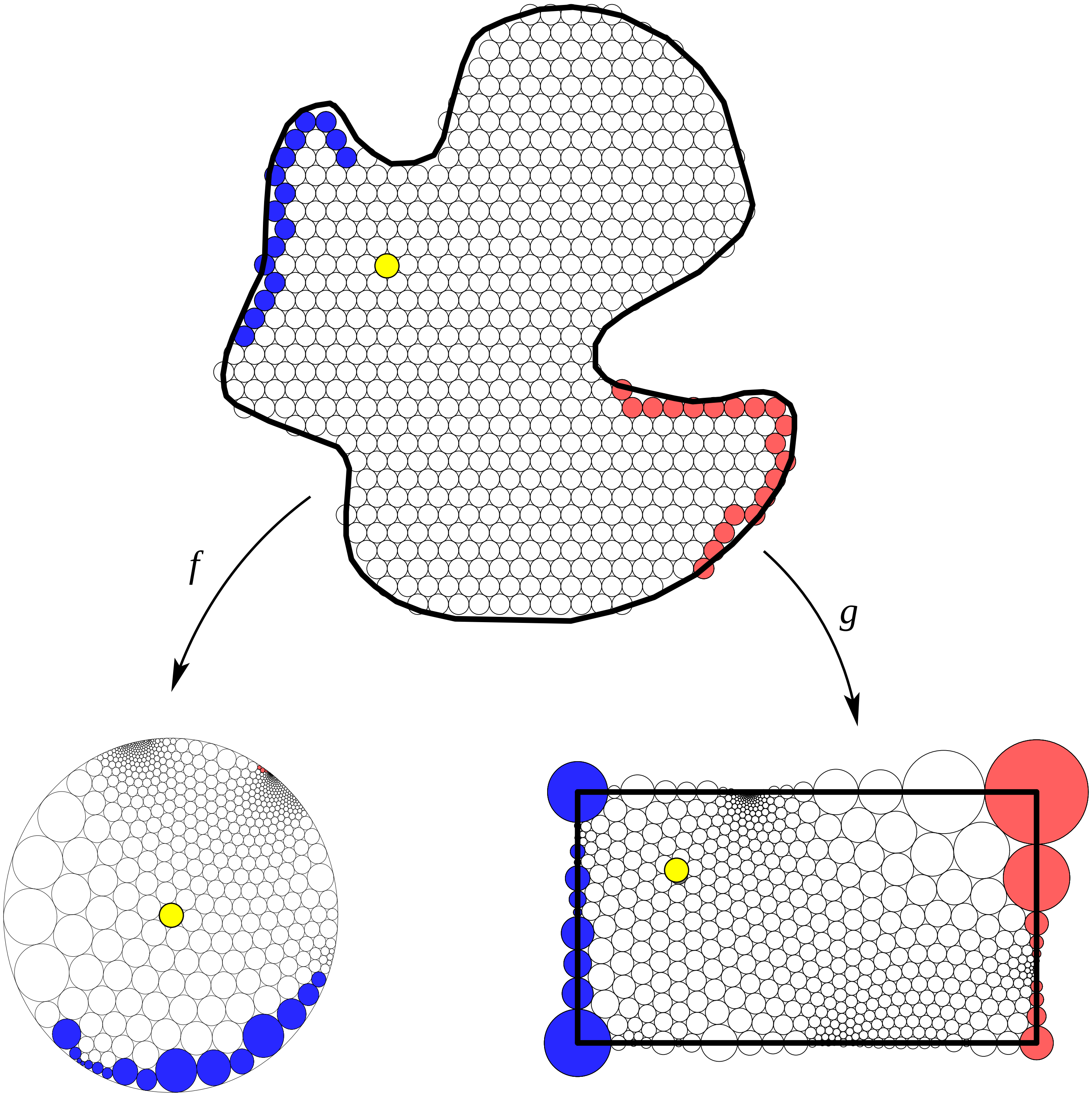
Discretized Versions

- Discrete conformal maps
- Random walks
- Discrete harmonic measure
- Discrete extremal length



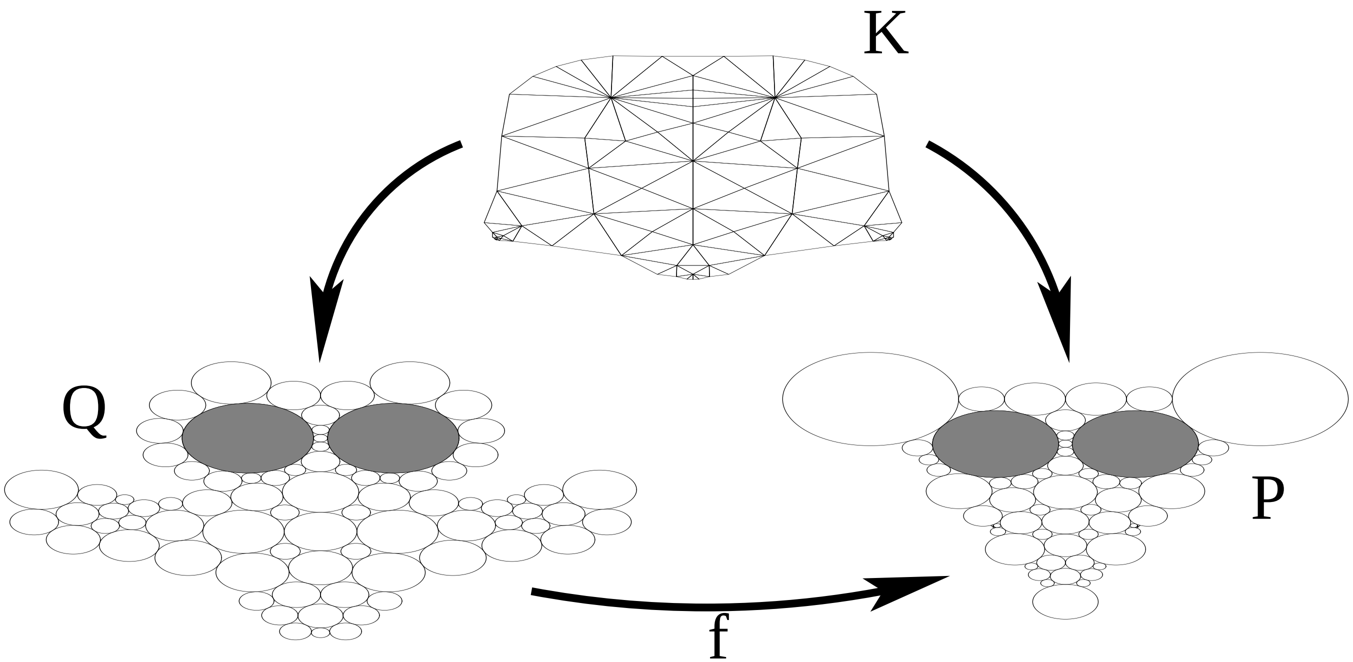



f


f
g

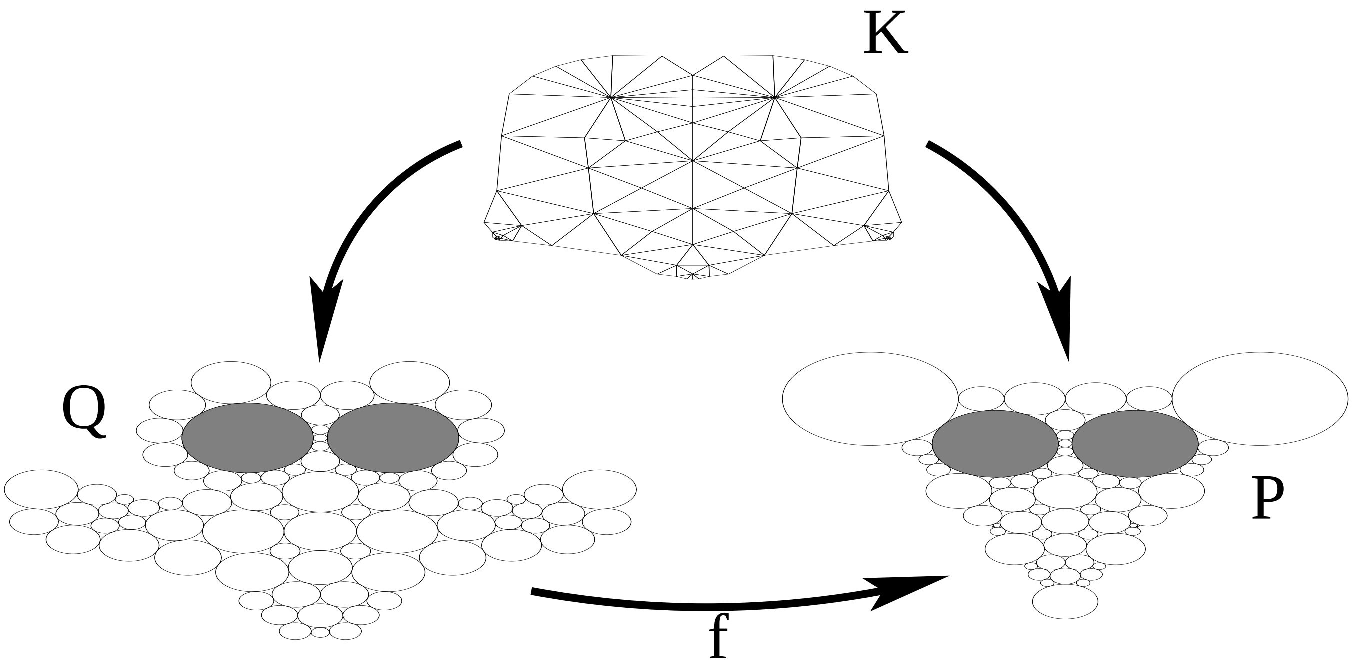
Discrete Conformal Mappings

Definition: A **discrete conformal mapping** is a map $f : Q \longrightarrow P$ between univalent circle packings associated with the same complex K.



Discrete Conformal Mappings

Definition: A **discrete conformal mapping** is a map $f : Q \longrightarrow P$ between univalent circle packings associated with the same complex K.

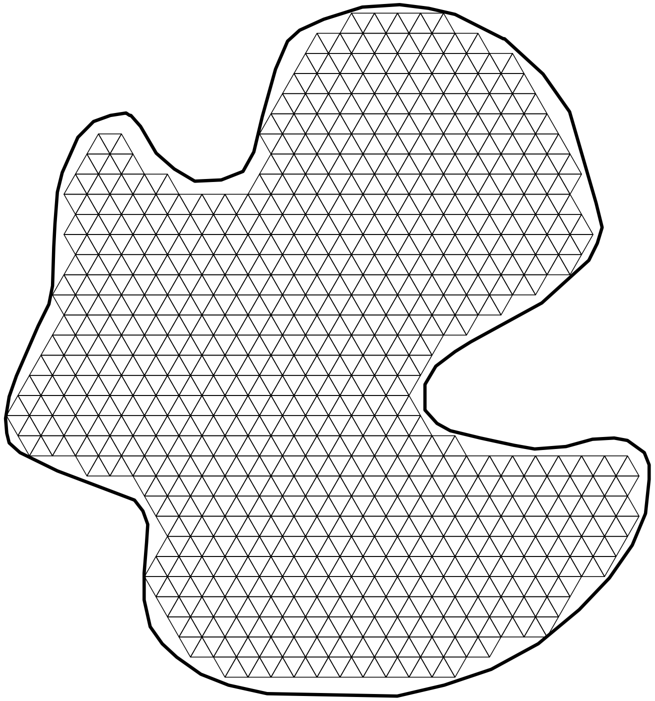


Proposal: A **discrete conformal structure** for an oriented topological surface $\mathcal{S}$ is a simplicial complex K which triangulates $\mathcal{S}$.

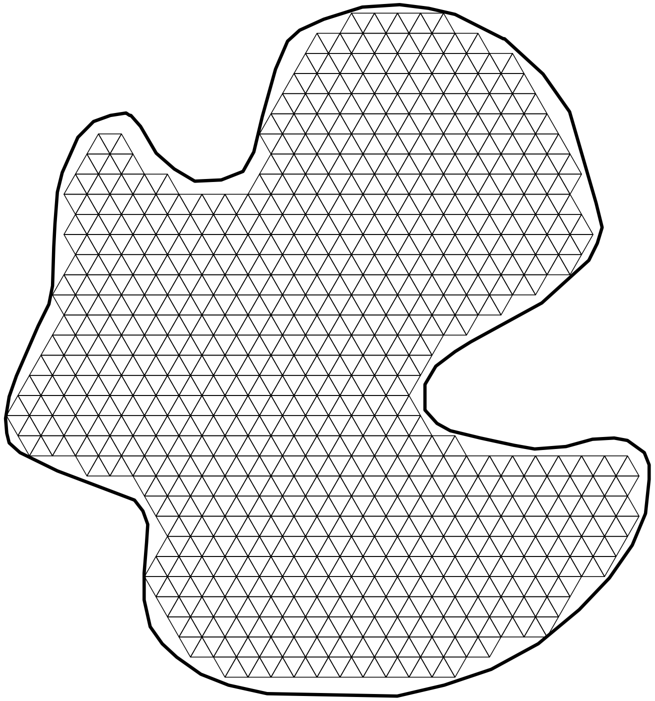
3. Emergent Conformal Structure

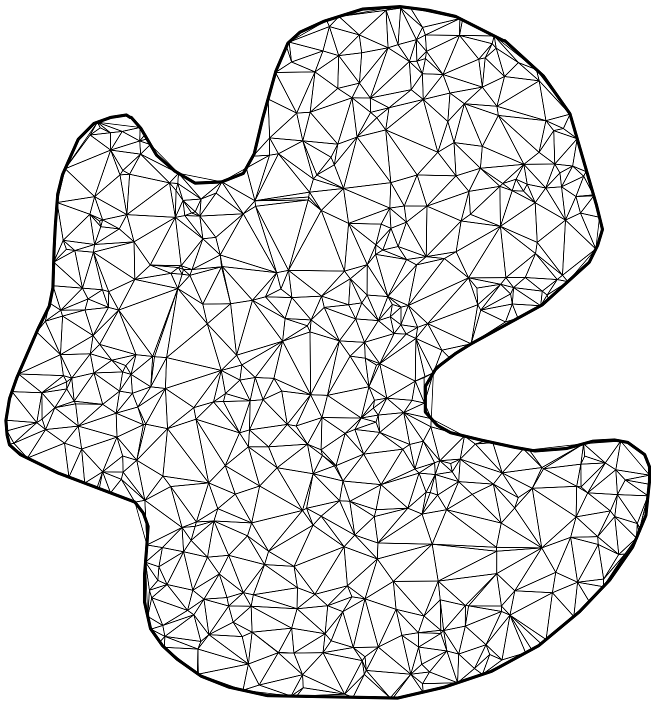
- A random idea
- Experimental support
- Intuition
- What is a “random” triangulation

Packing Triangulations



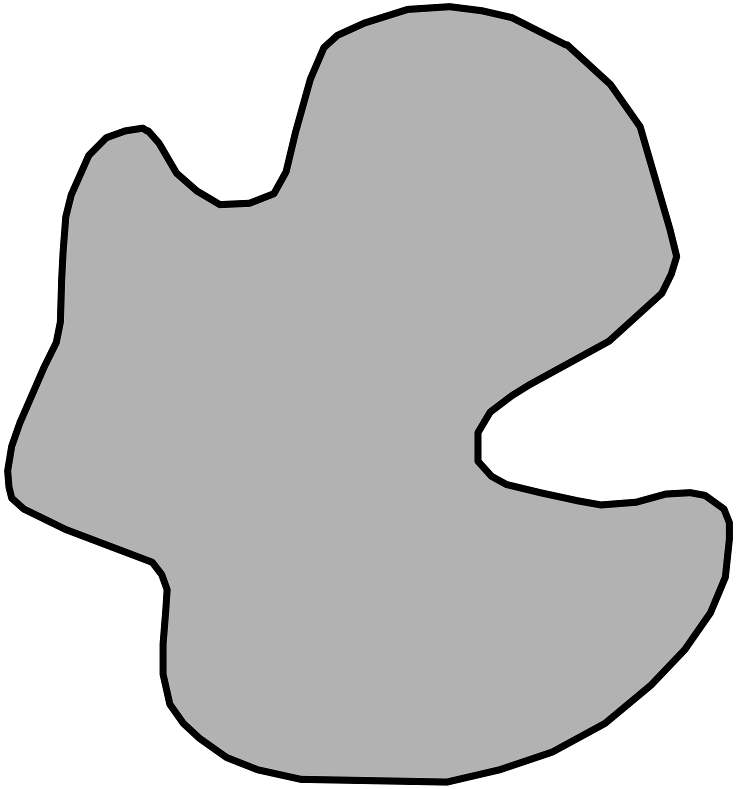
Packing Triangulations ⟶ Random Triangulations

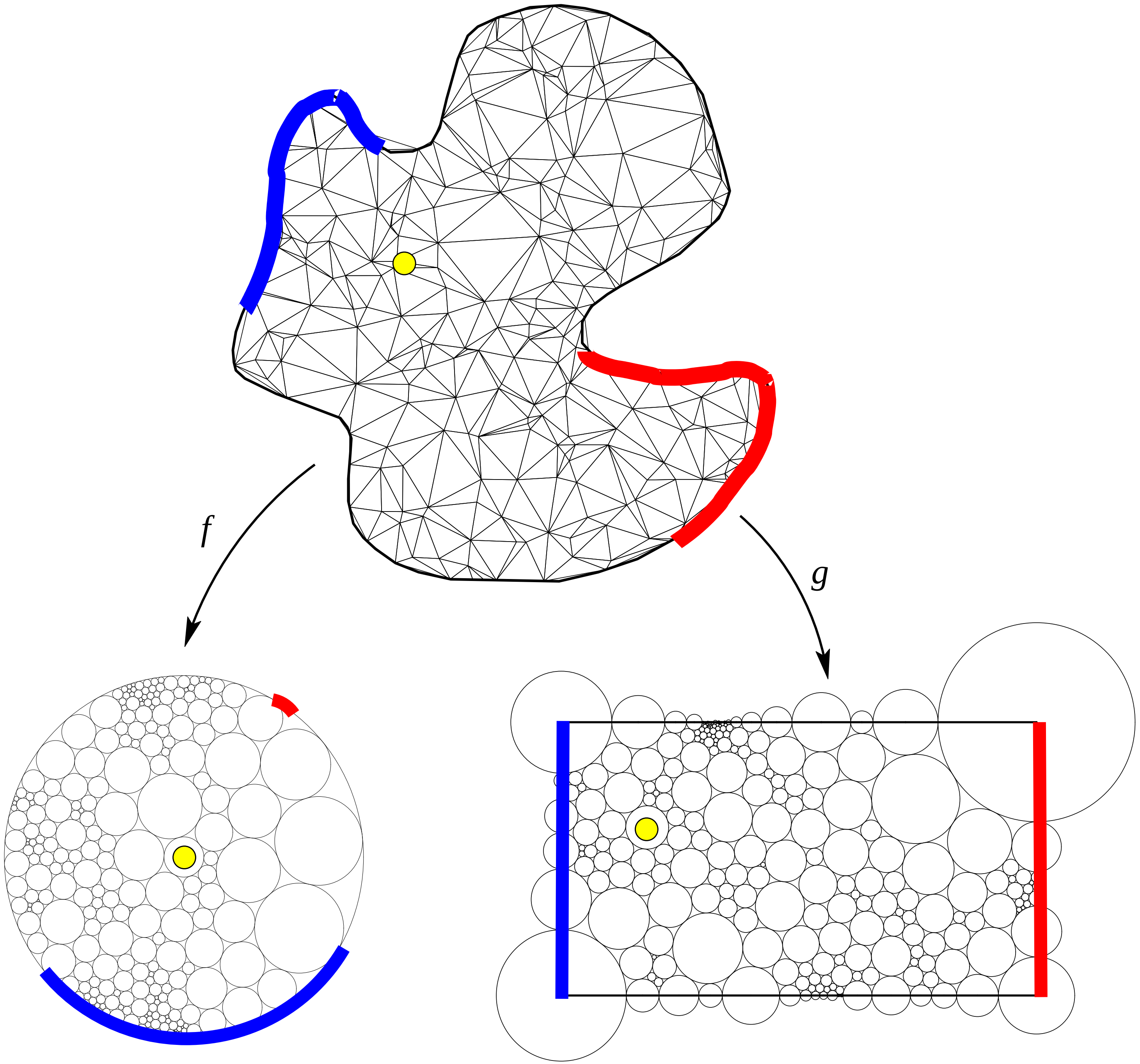




Random Triangulations — and Companions

- Random discrete maps
- Random walks
- Discrete harmonic measure
- Discrete extremal length




f
g

Emergent Conformal Structure

Setting: Let Ω be a bounded simply connected plane domain, $z_1, z_2 \in \Omega$, and let $F : \Omega \longrightarrow \mathbb{D}$ be the unique conformal mapping with $F(z_1) = 0, F(z_2) > 0$.

Emergent Conformal Structure

Setting: Let Ω be a bounded simply connected plane domain, $z_1, z_2 \in \Omega$, and let $F : \Omega \longrightarrow \mathbb{D}$ be the unique conformal mapping with $F(z_1) = 0, F(z_2) > 0$.

Random Maps: For $n >> 1$, define a "random" map $f_n : \Omega \longrightarrow \mathbb{D}$ as follows:

- Select a random triangulation K_n of Ω having n vertices
- Compute the maximal circle packing P_n for K_n (in $\mathbb{D}$)
- Define $f_n : K_n \longrightarrow$ carrier(P_n) (An appropriate $\phi \in$ Auto$(\mathbb{D})$ applied to P_n ensures $f_n(z_1) = 0,\ f_n(z_2) > 0$)

Emergent Conformal Structure

Setting: Let Ω be a bounded simply connected plane domain, $z_1, z_2 \in \Omega$, and let $F : \Omega \longrightarrow \mathbb{D}$ be the unique conformal mapping with $F(z_1) = 0, F(z_2) > 0$.

Random Maps: For $n >> 1$, define a "random" map $f_n : \Omega \longrightarrow \mathbb{D}$ as follows:

- Select a random triangulation K_n of Ω having n vertices
- Compute the maximal circle packing P_n for K_n (in $\mathbb{D}$)
- Define $f_n : K_n \longrightarrow$ carrier(P_n) (An appropriate $\phi \in$ Auto$(\mathbb{D})$ applied to P_n ensures $f_n(z_1) = 0,\ f_n(z_2) > 0$)

Conjecture: When Ω, F, and f_n are as above, then $f_n \xrightarrow{P} F$ as $n \to \infty$; that is, the random maps converge "in probability" to the conformal map F.

Emergent Conformal Structure

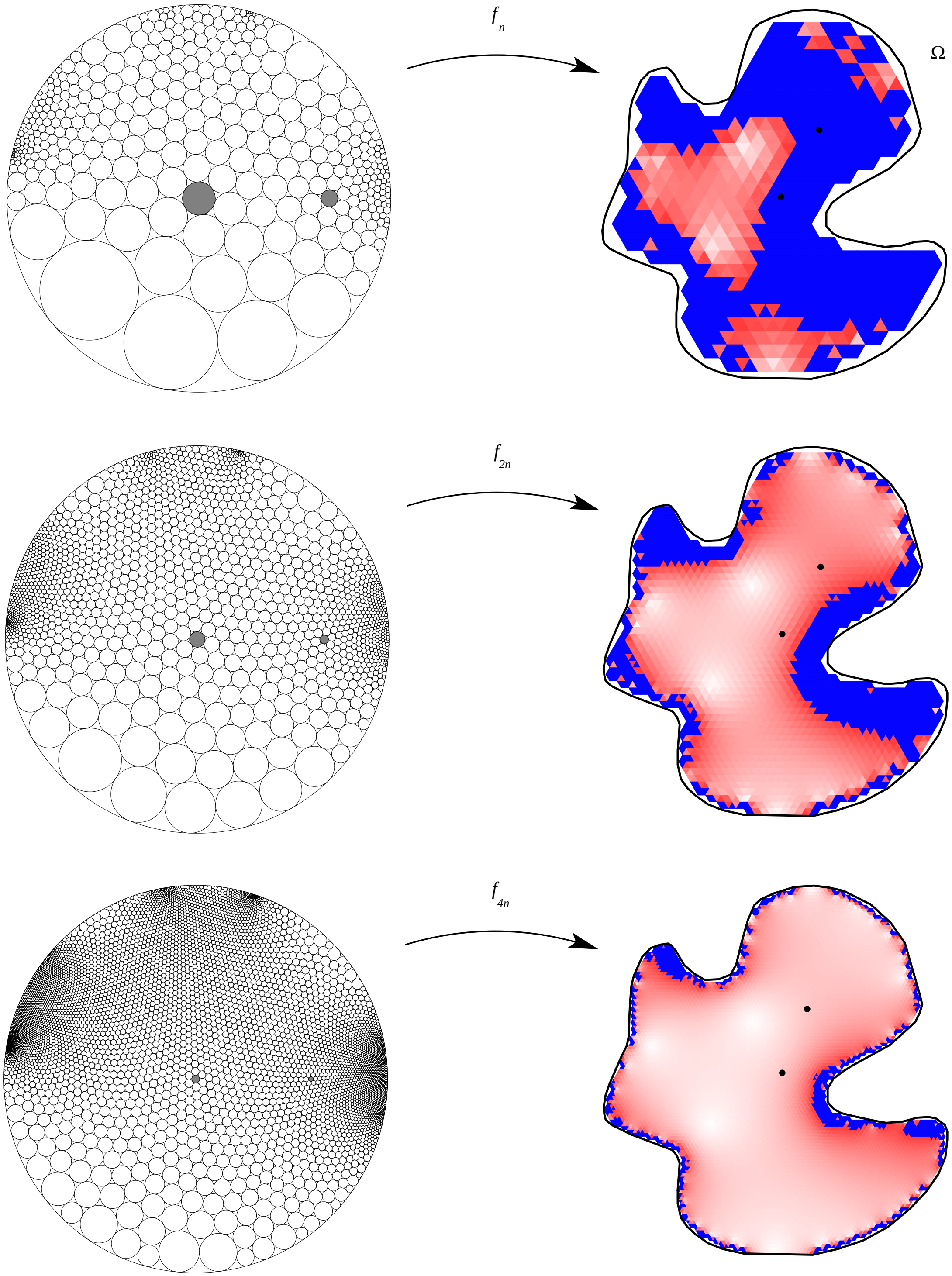
Setting: Let Ω be a bounded simply connected plane domain, $z_1, z_2 \in \Omega$, and let $F : \Omega \longrightarrow \mathbb{D}$ be the unique conformal mapping with $F(z_1) = 0, F(z_2) > 0$.

Random Maps: For $n >> 1$, define a "random" map $f_n : \Omega \longrightarrow \mathbb{D}$ as follows:

- Select a random triangulation K_n of Ω having n vertices
- Compute the maximal circle packing P_n for K_n (in $\mathbb{D}$)
- Define $f_n : K_n \longrightarrow$ carrier(P_n) (An appropriate $\phi \in$ Auto$(\mathbb{D})$ applied to P_n ensures $f_n(z_1) = 0,\ f_n(z_2) > 0$)

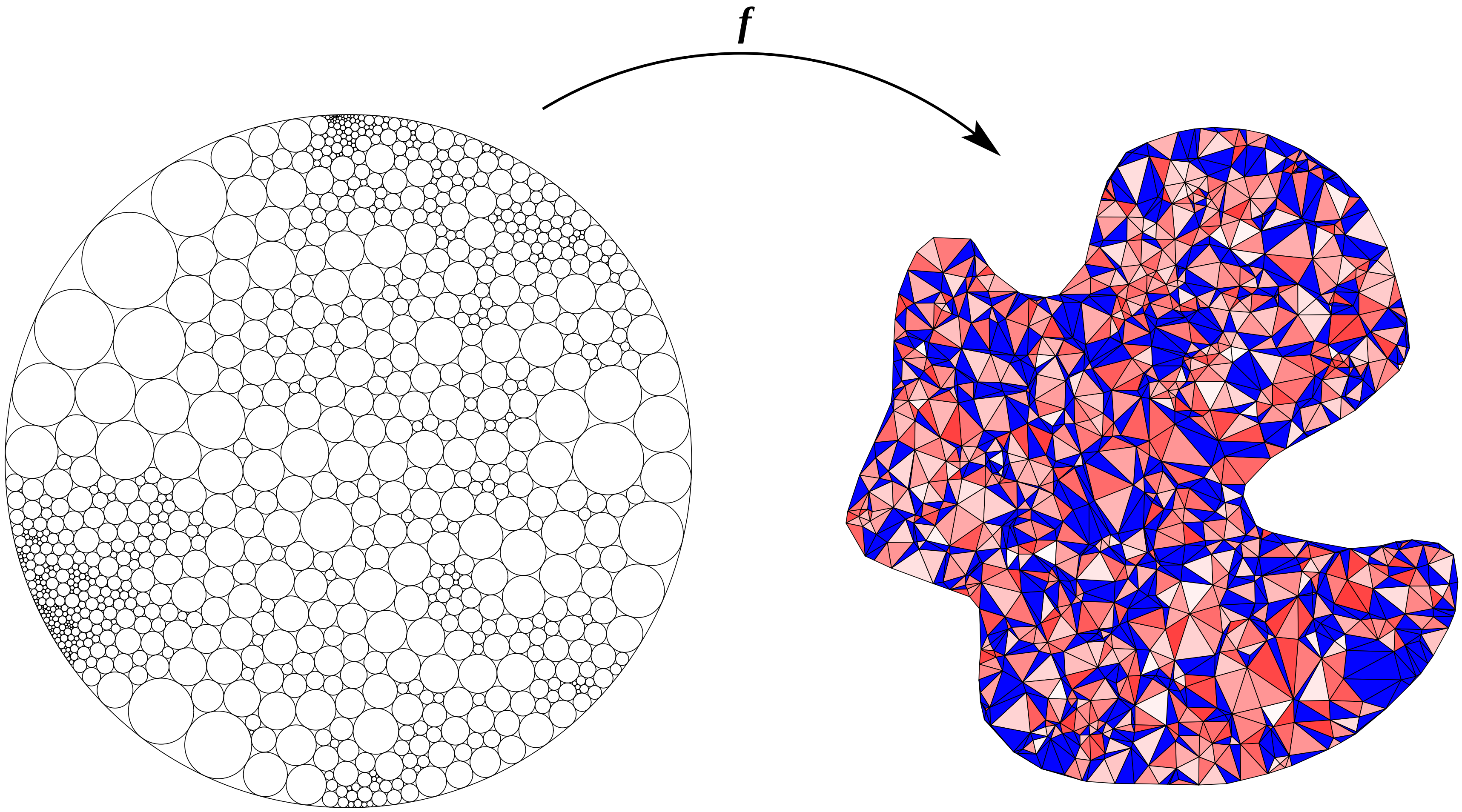
Conjecture: When Ω, F, and f_n are as above, then $f_n \xrightarrow{P} F$ as $n \to \infty$; that is, the random maps converge "in probability" to the conformal map F.

Speculation: This should extend readily to general Riemann surfaces for an appropriate notion of "random triangulation".



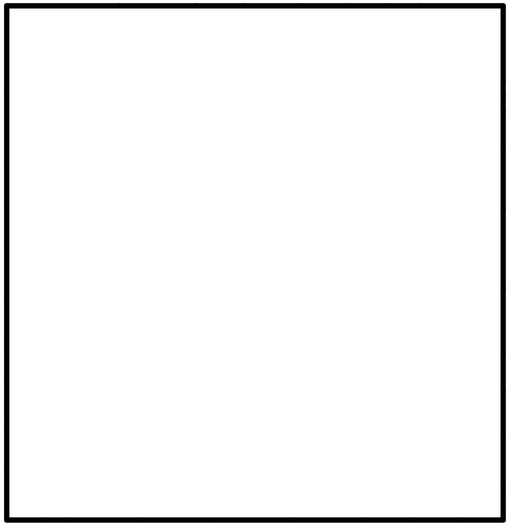
f_n
Ω
f_{2n}
f_{4n}

Distribution of Dilatations

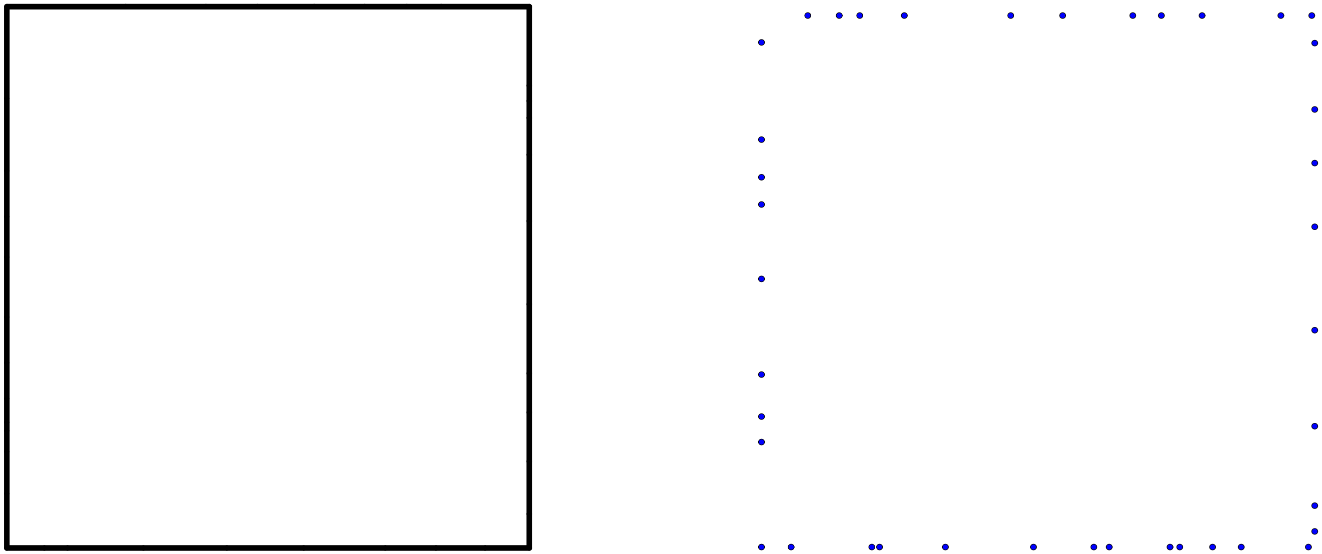


Color coding by qc-dilatation k; faces with dilatation $k > 2$ are blue.

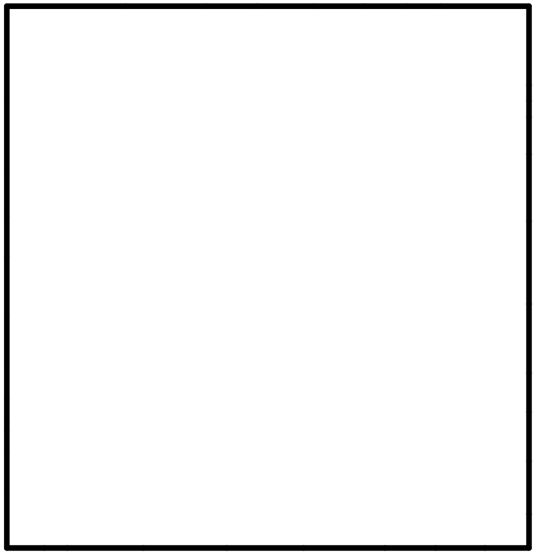
Random Square Construction

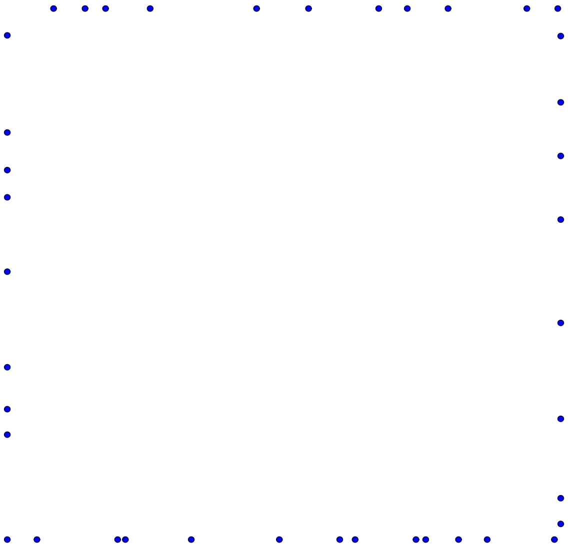


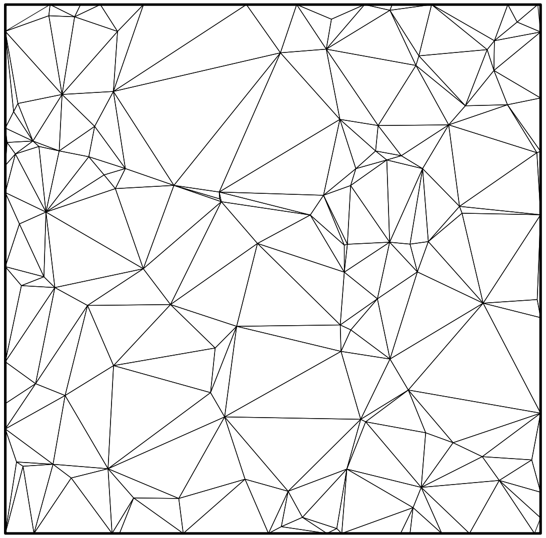
Random Square Construction



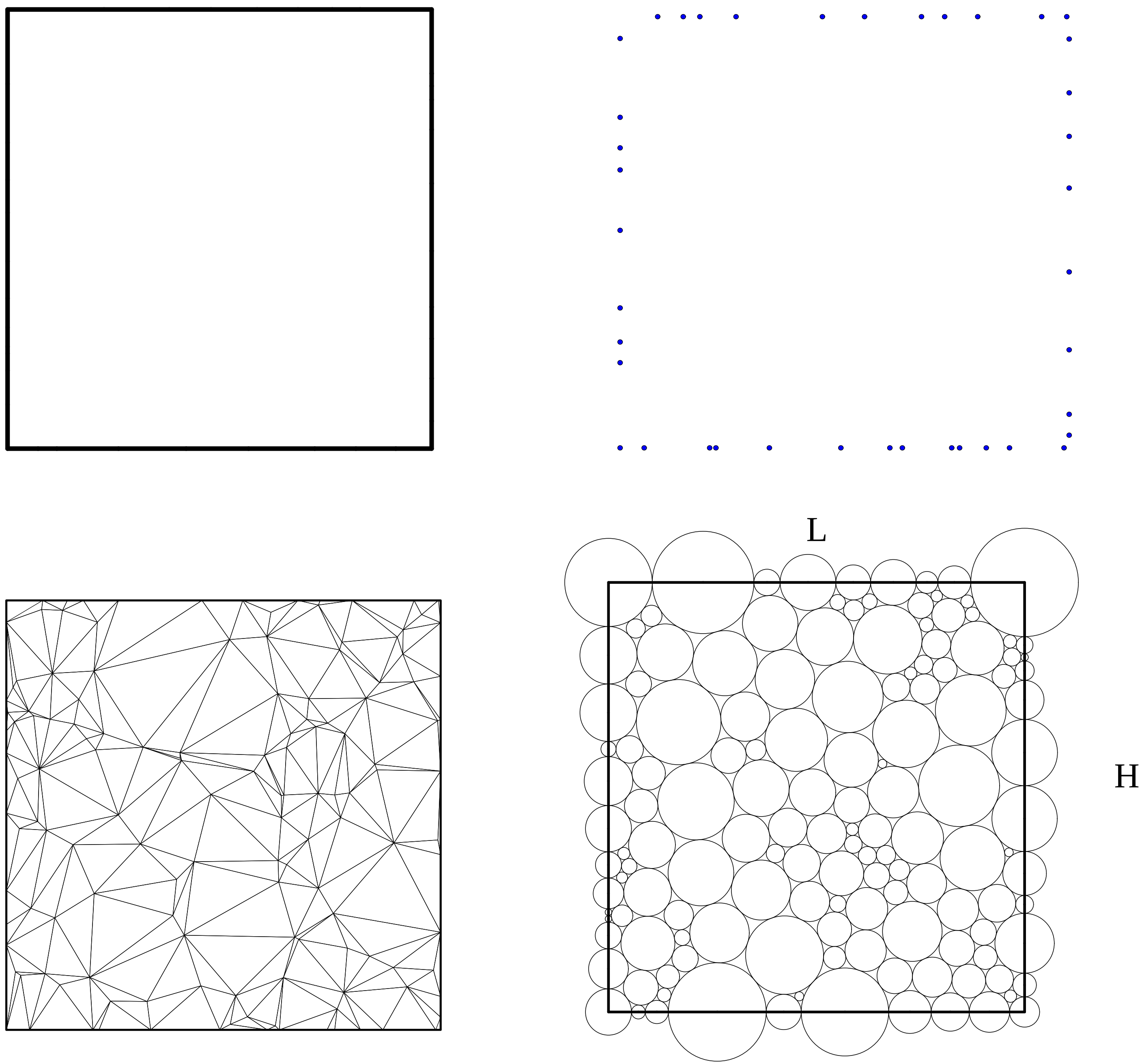
Random Square Construction







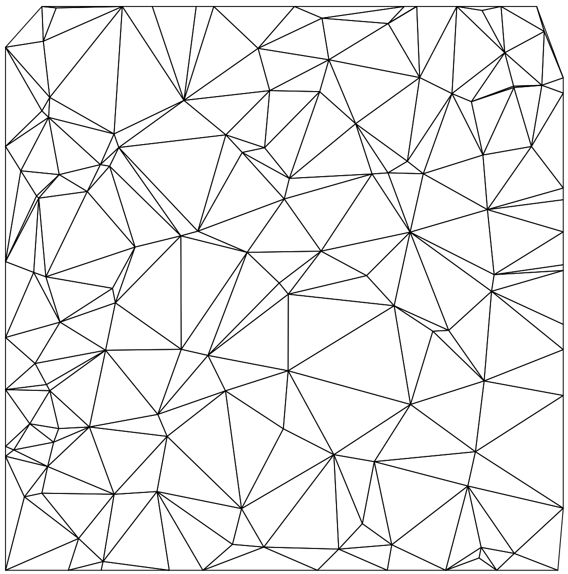
Random Square Construction



log (aspect)=log(H/L)

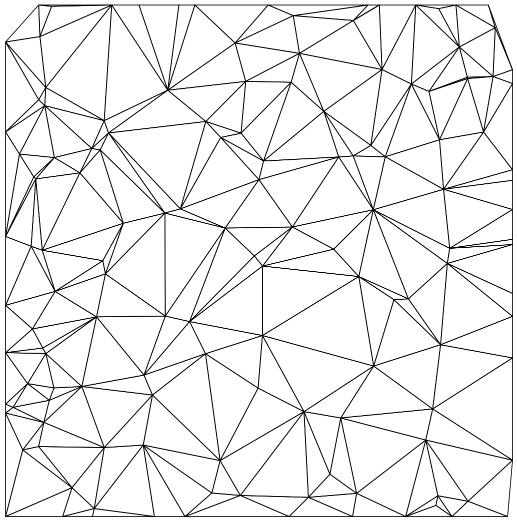
Square Map Detail

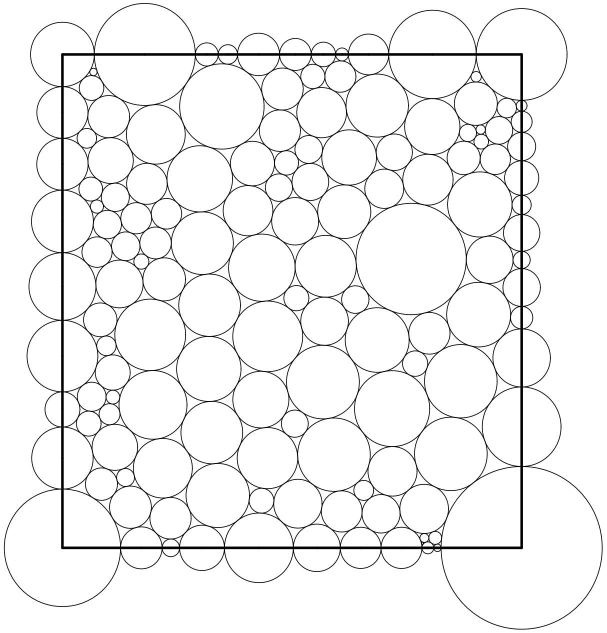
Create random K



Square Map Detail

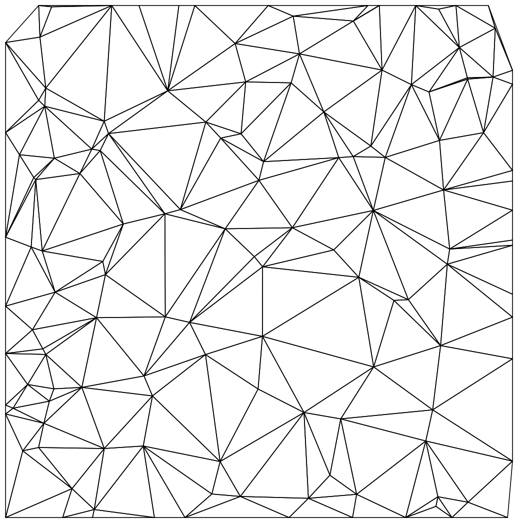
Create random $K \longrightarrow$ circle pack it

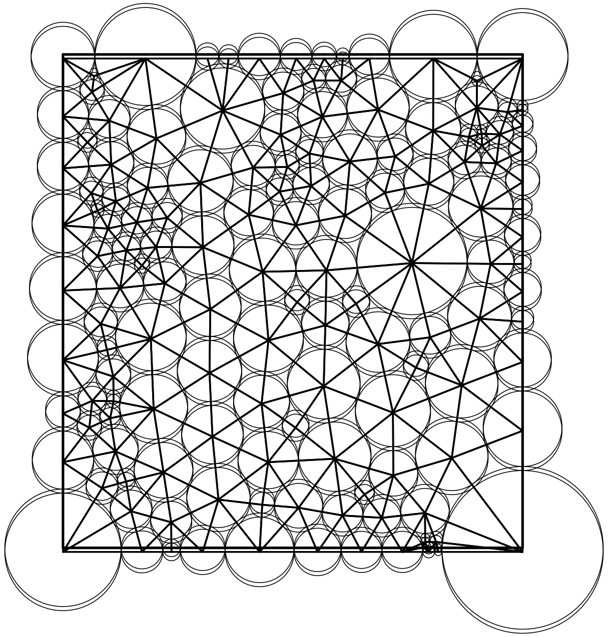




Square Map Detail

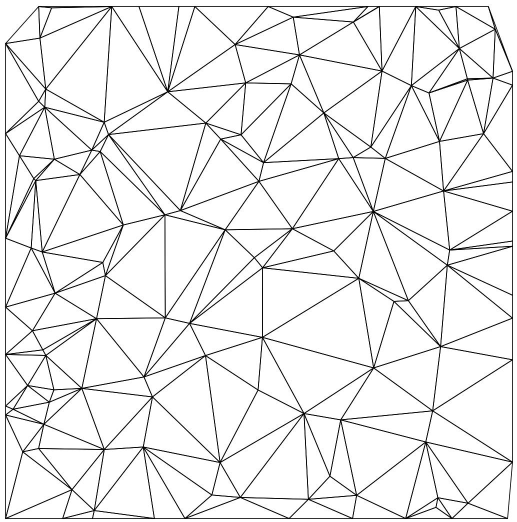
Create random $K \longrightarrow$ circle pack it $\longrightarrow$ the carrier is equivalent to K

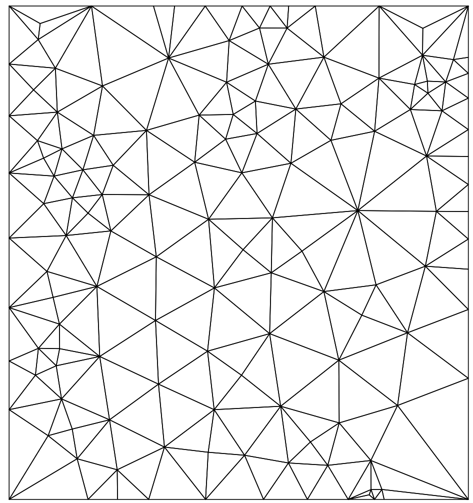




Square Map Detail

Create random $K \longrightarrow$ circle pack it $\longrightarrow$ the carrier is equivalent to K

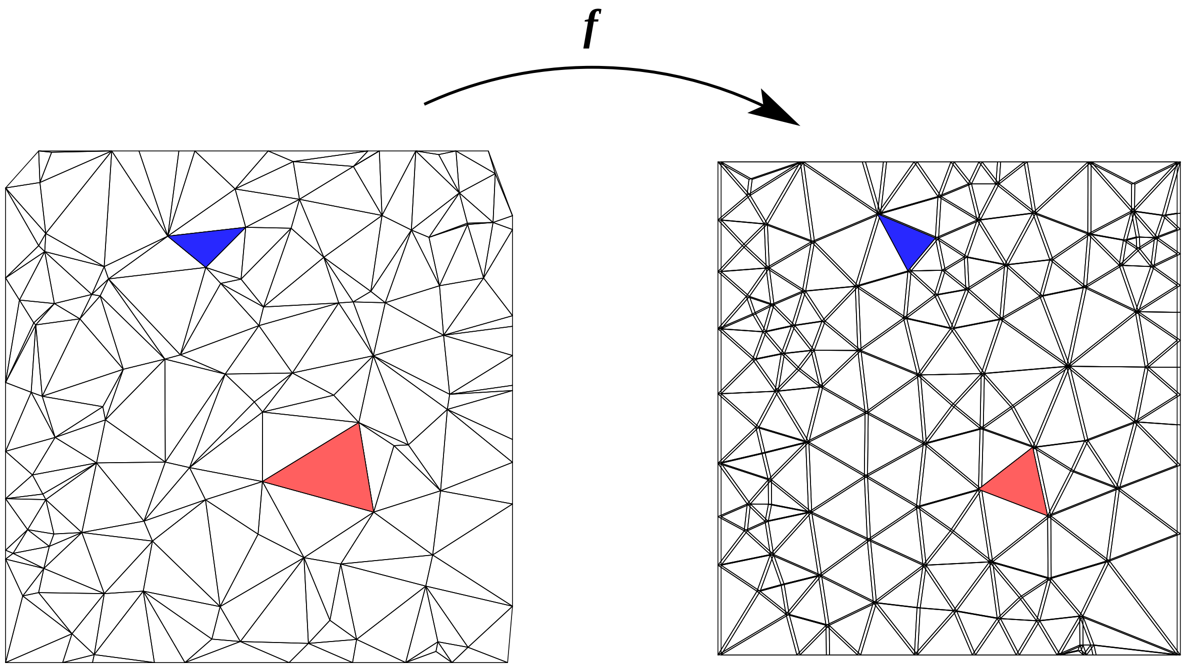




Disregard the circles, leaving the “carrier”.

Square Map Detail

Create random $K \longrightarrow$ circle pack it $\longrightarrow$ the carrier is equivalent to K

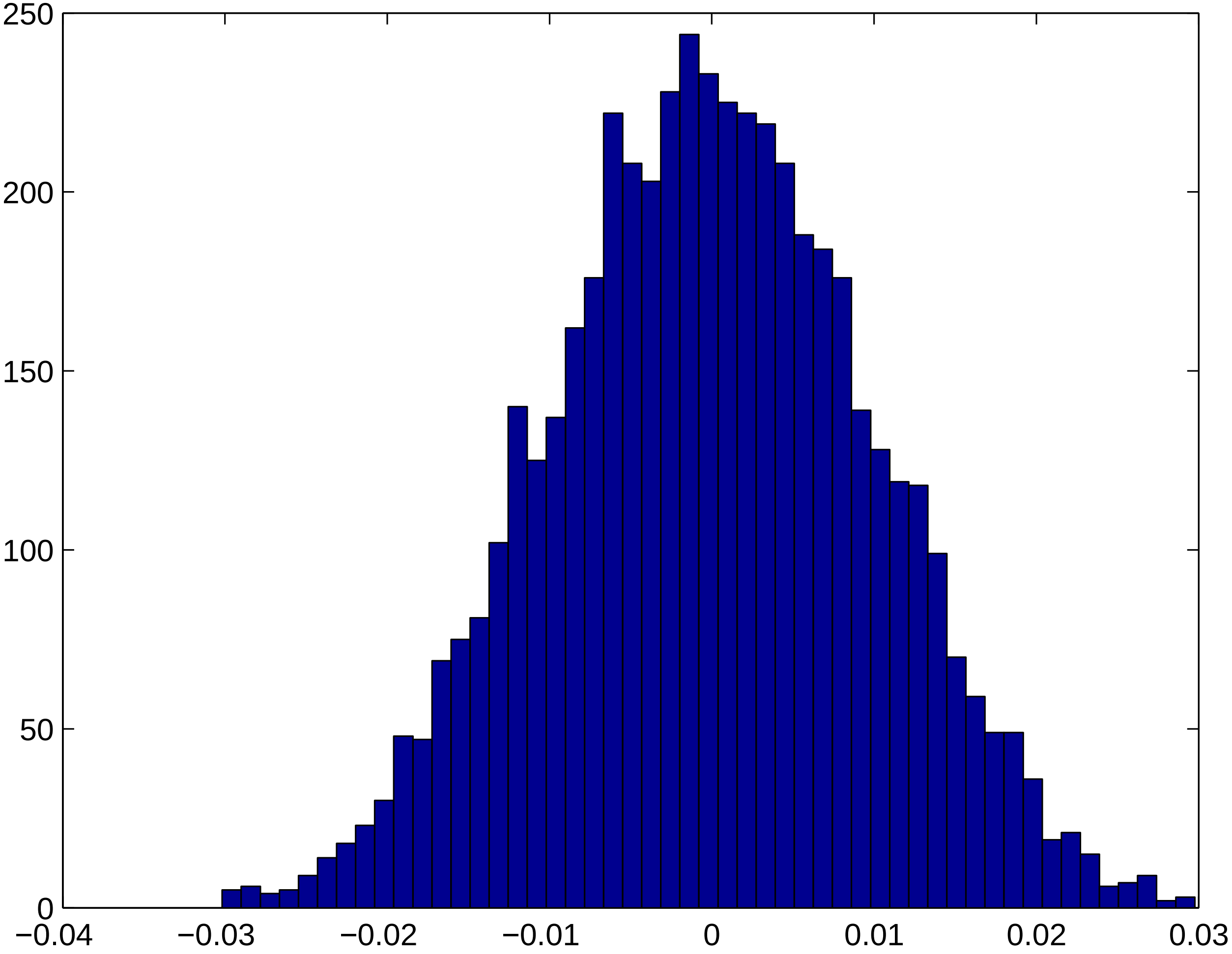


Disregard the circles, leaving the “carrier”.

The map f is a piecewise affine map between the random triangulation and the “carrier” of the circle packing.

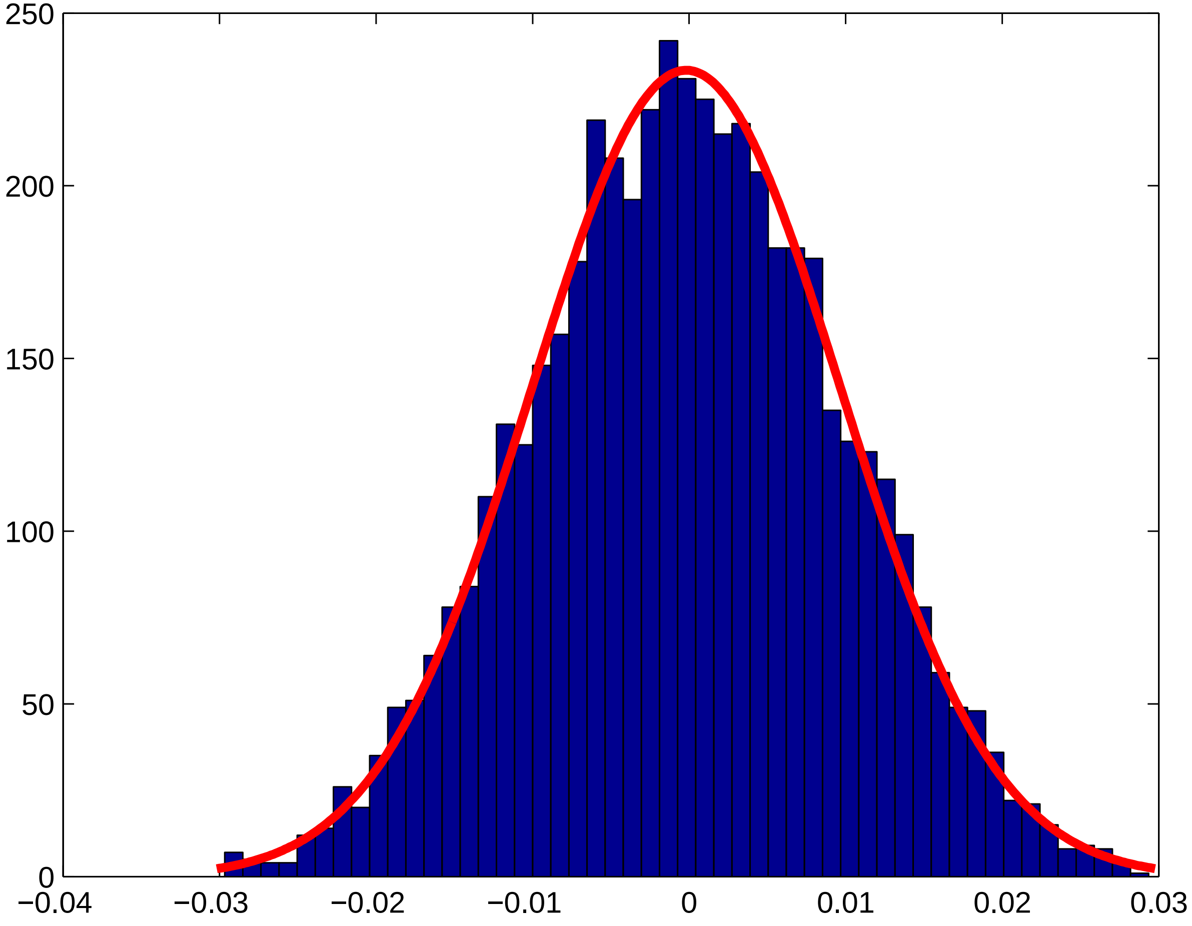
Experiments with the Square

5000 trials with 3200 random vertices per trial yield this histogram:



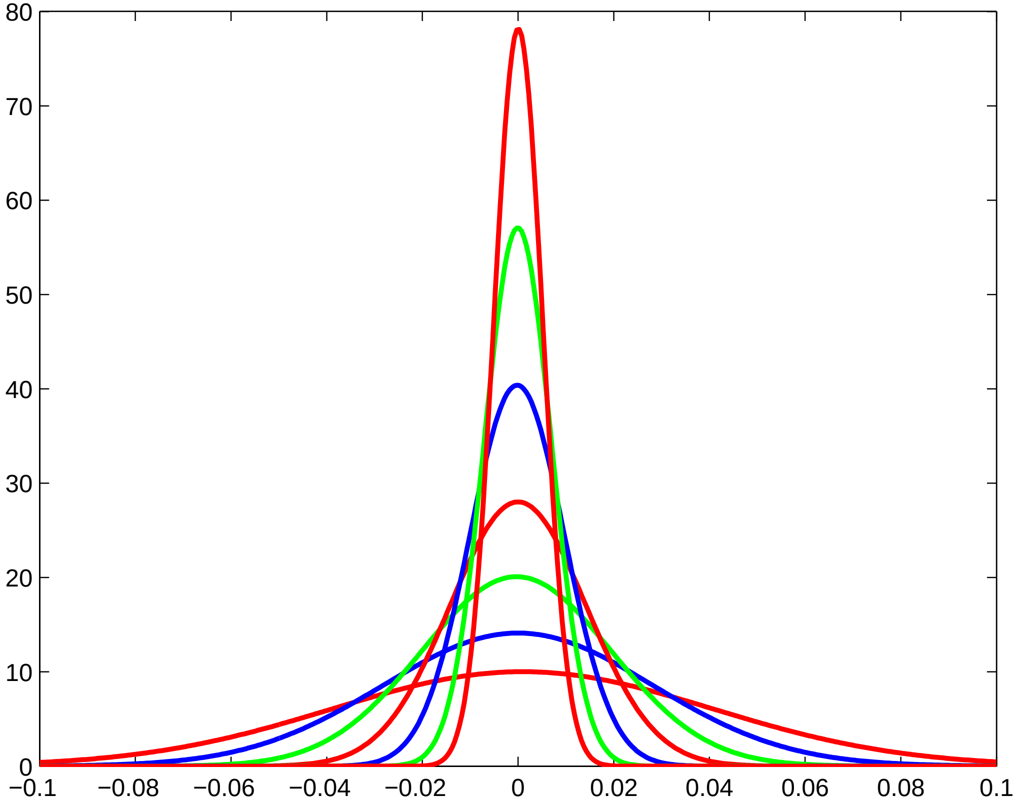
Experiments with the Square

5000 trials with 3200 random vertices per trial yield this histogram:



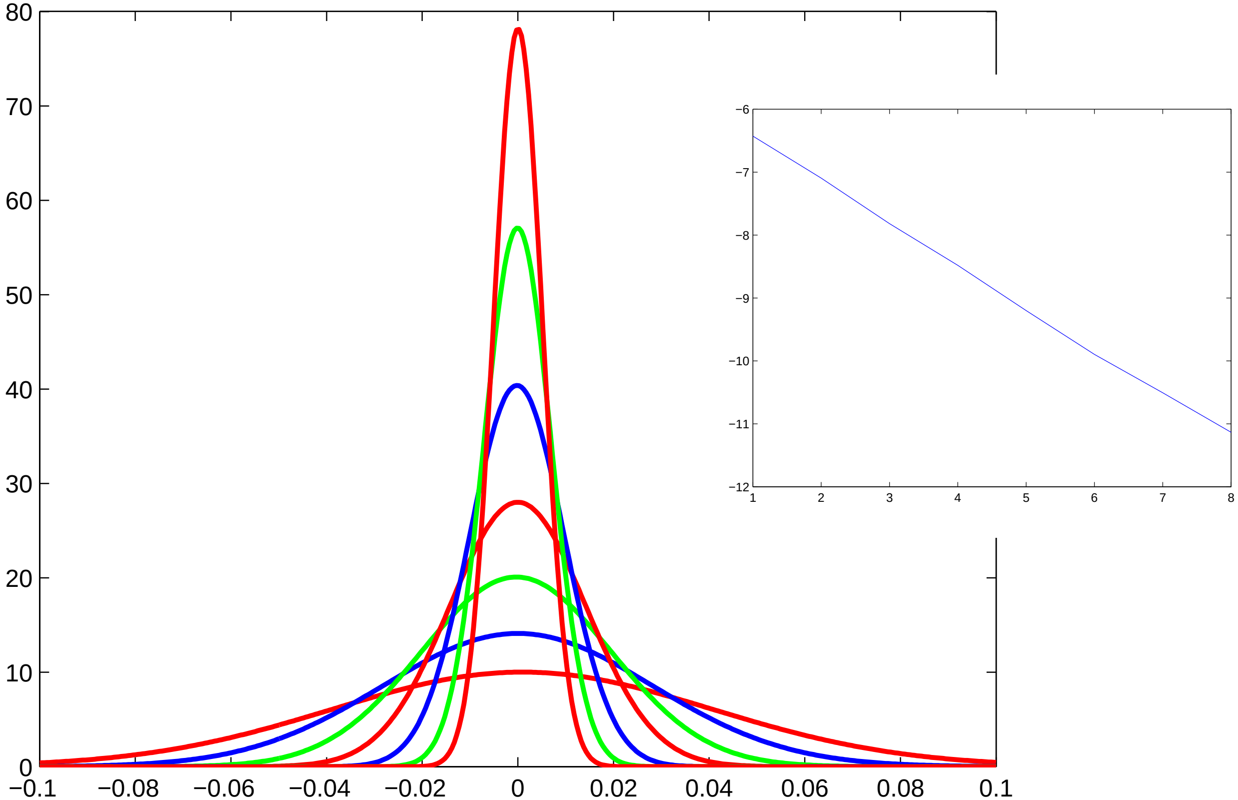
Visually and with QQ-plot the distribution appears to be gaussian.

Varying the Complexity



Here are plots, 5000 trials each for $N = 200, 400, 800, 1600, 3200, 6400, 12800$.

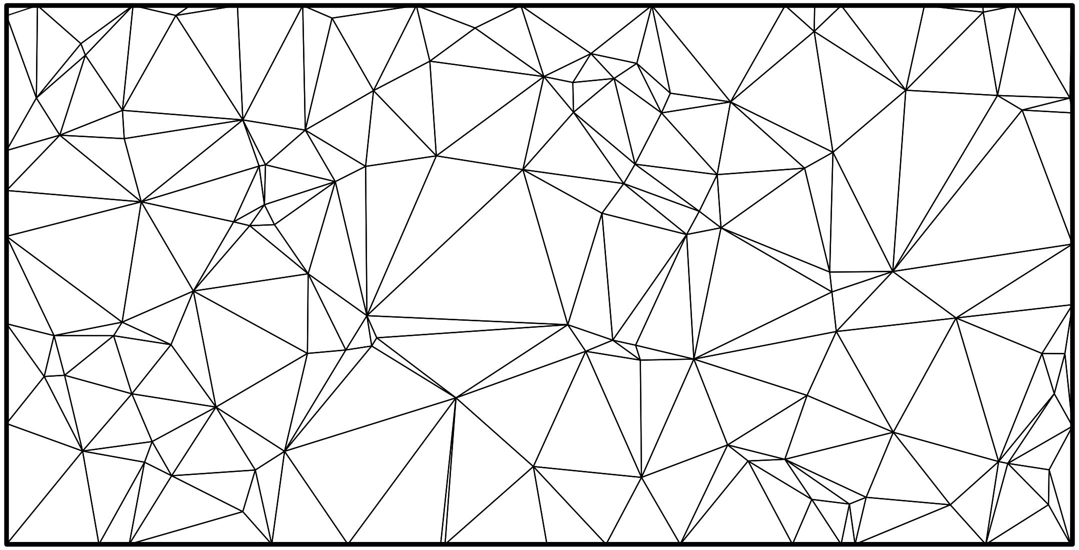
Varying the Complexity



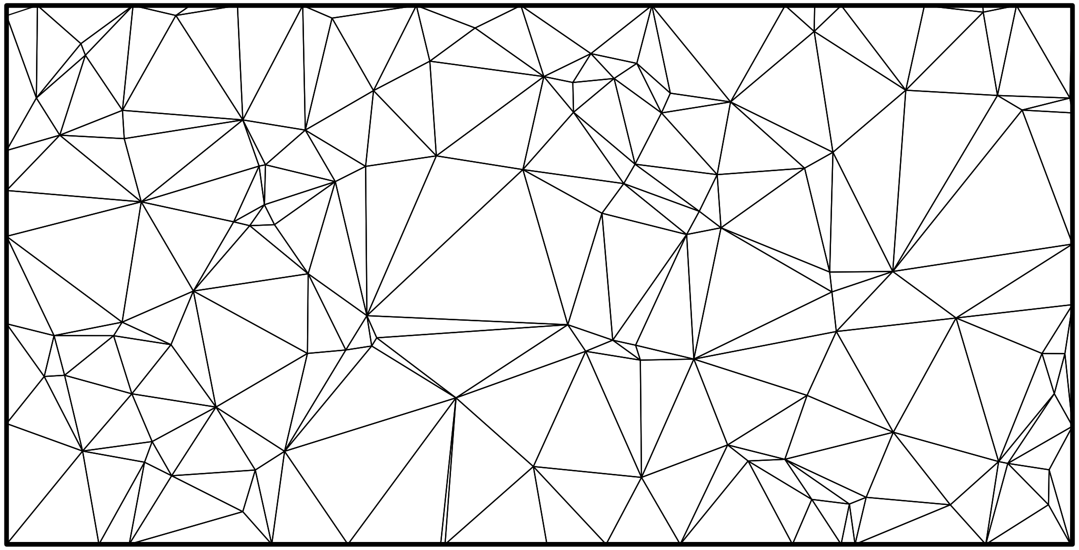
Here are plots, 5000 trials each for $N = 200, 400, 800, 1600, 3200, 6400, 12800$. A log-log plot of variance shows:

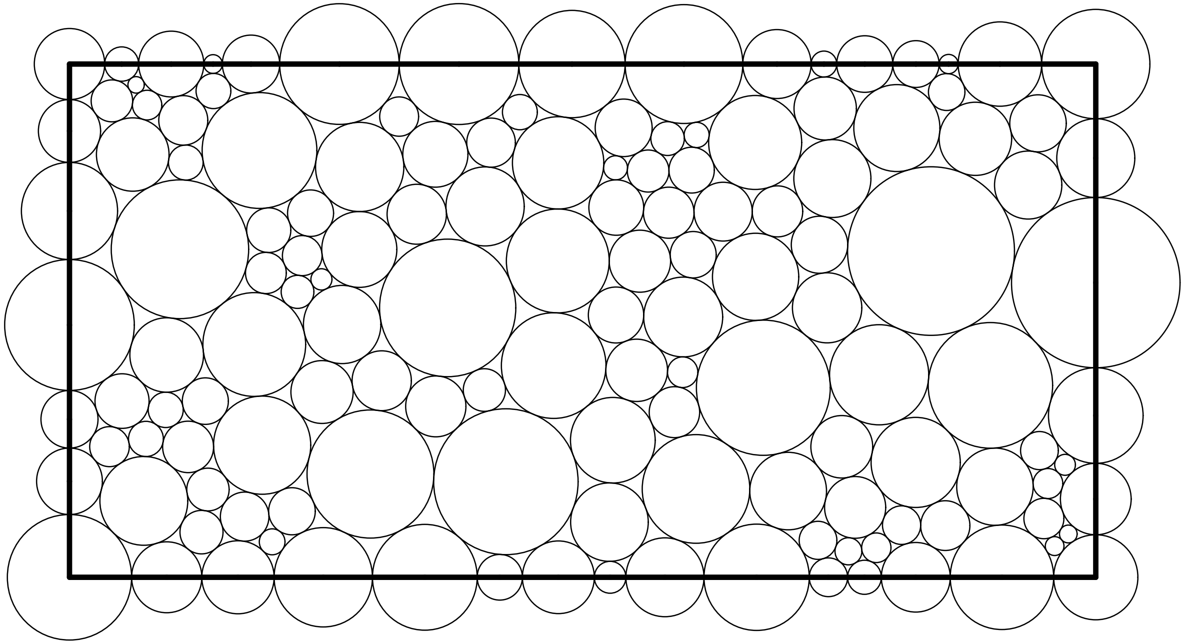
"double N and you halve the variance."

A Rectangles of Aspect 2



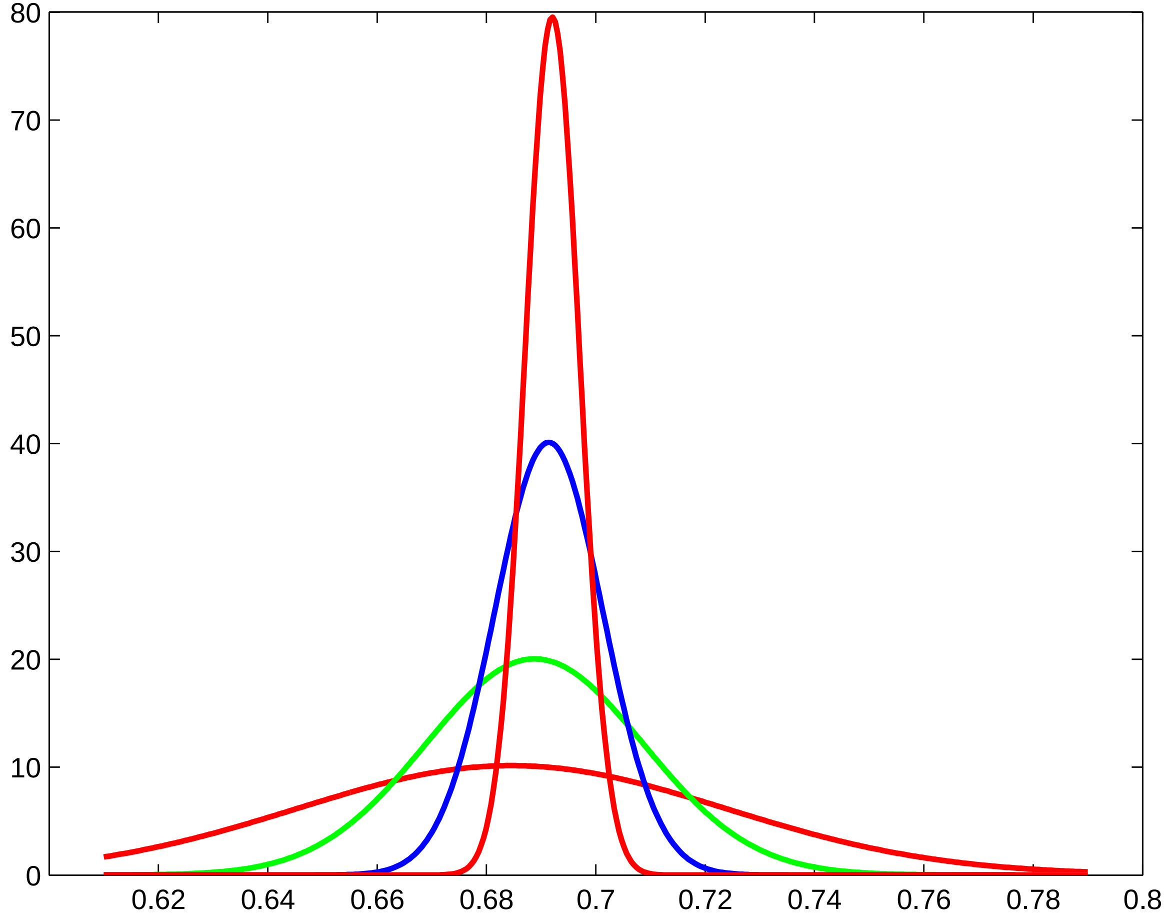
A Rectangles of Aspect 2



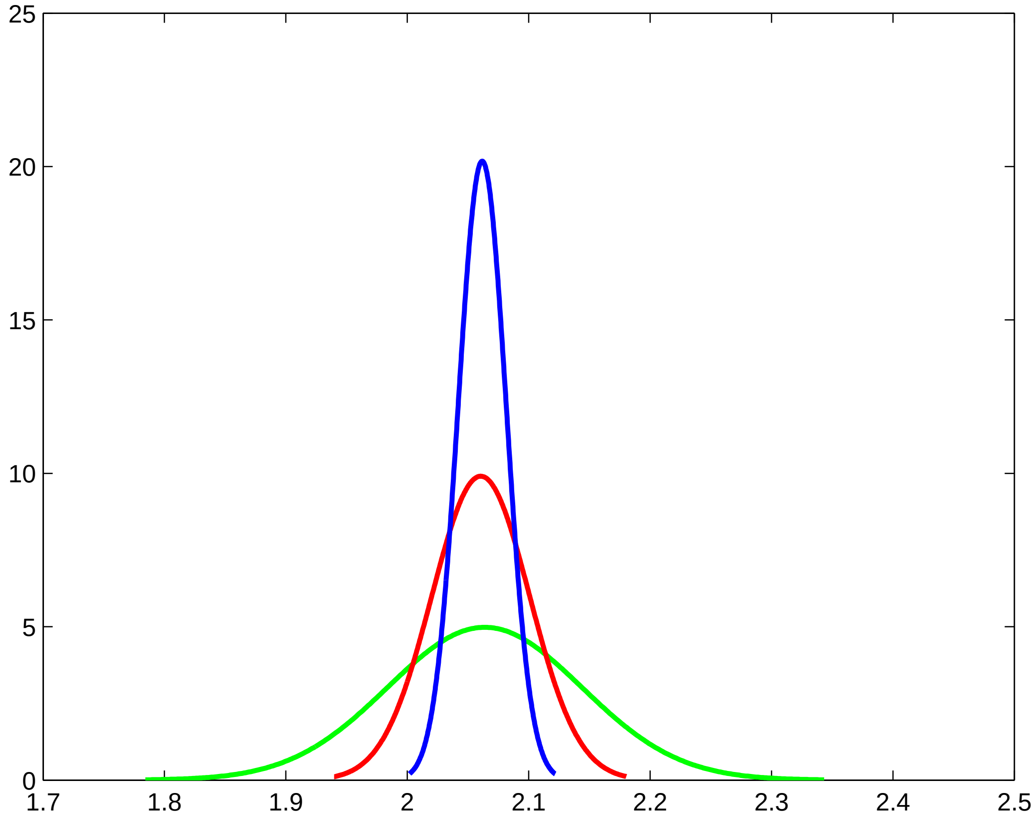


Trials for Aspect 2

5000 random trials each for $N = 200, 800, 3200, 12800$. (Truth $\log(2) \approx 0.6931$)



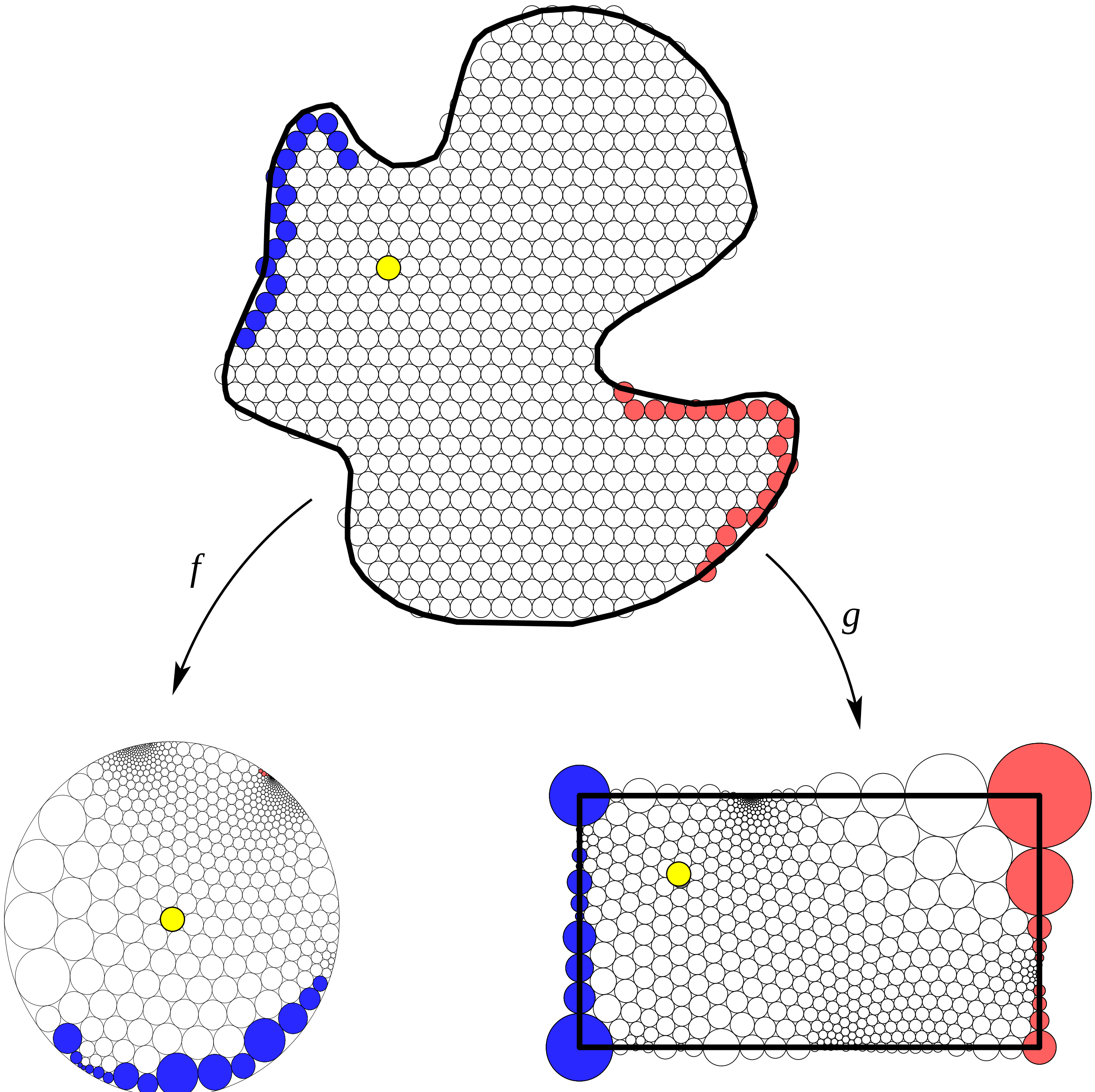
Torus Triangulations



5000 trials with various N for torus of modulus (1+4i)/2:

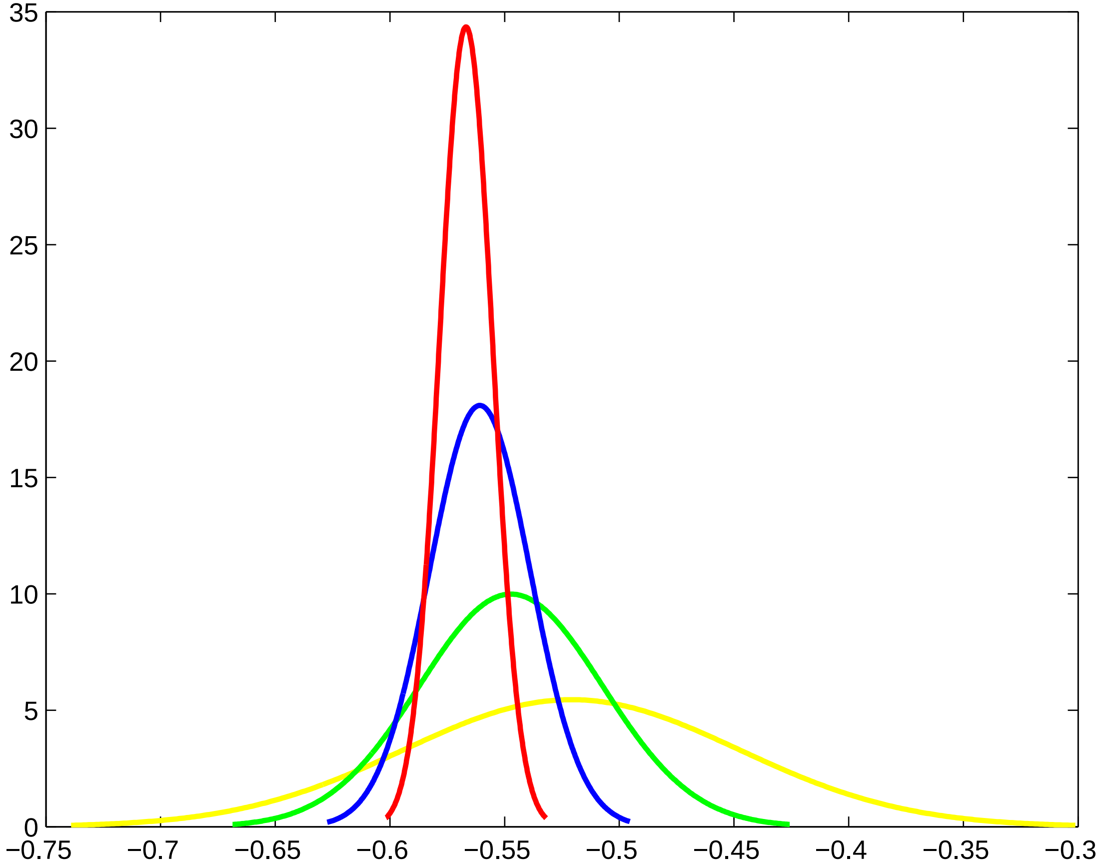
N	mean (true=2.0616)	variance
200	2.0638	.00642
800	2.0605	.00162
3200	2.0617	.00039

Back to Ω



Extremal Length Trials

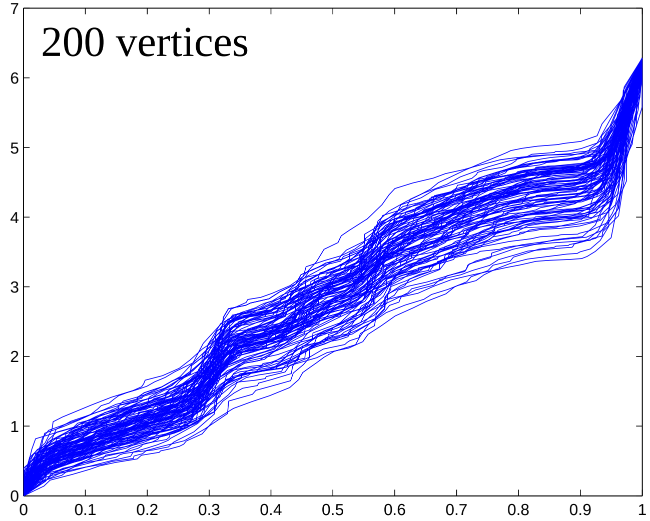
Measure extremal length of the paths between the red and blue arcs in $\partial\Omega$



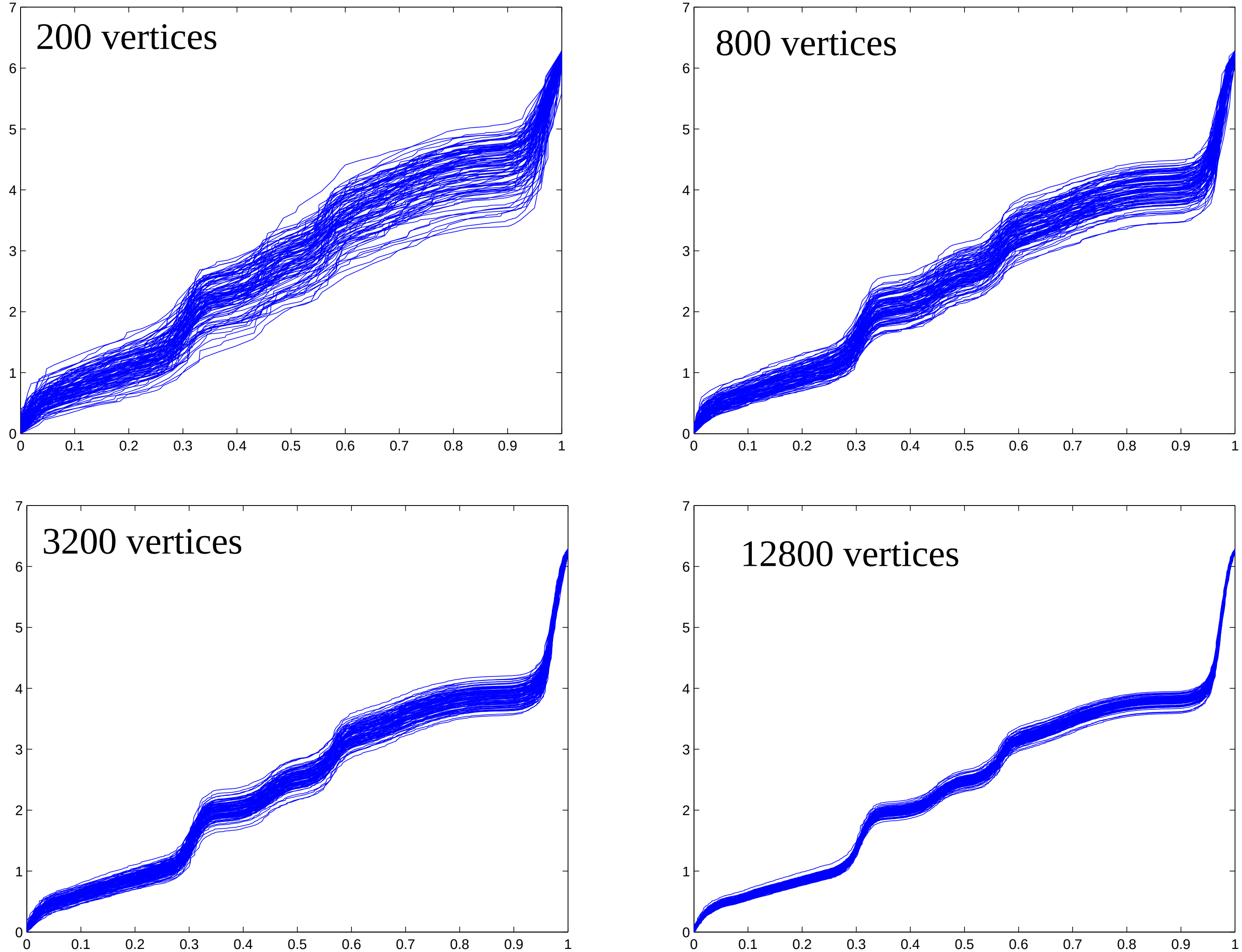
5000 trials each, $N = 200, 800, 3200, 12800$.

Harmonic Measure Trials

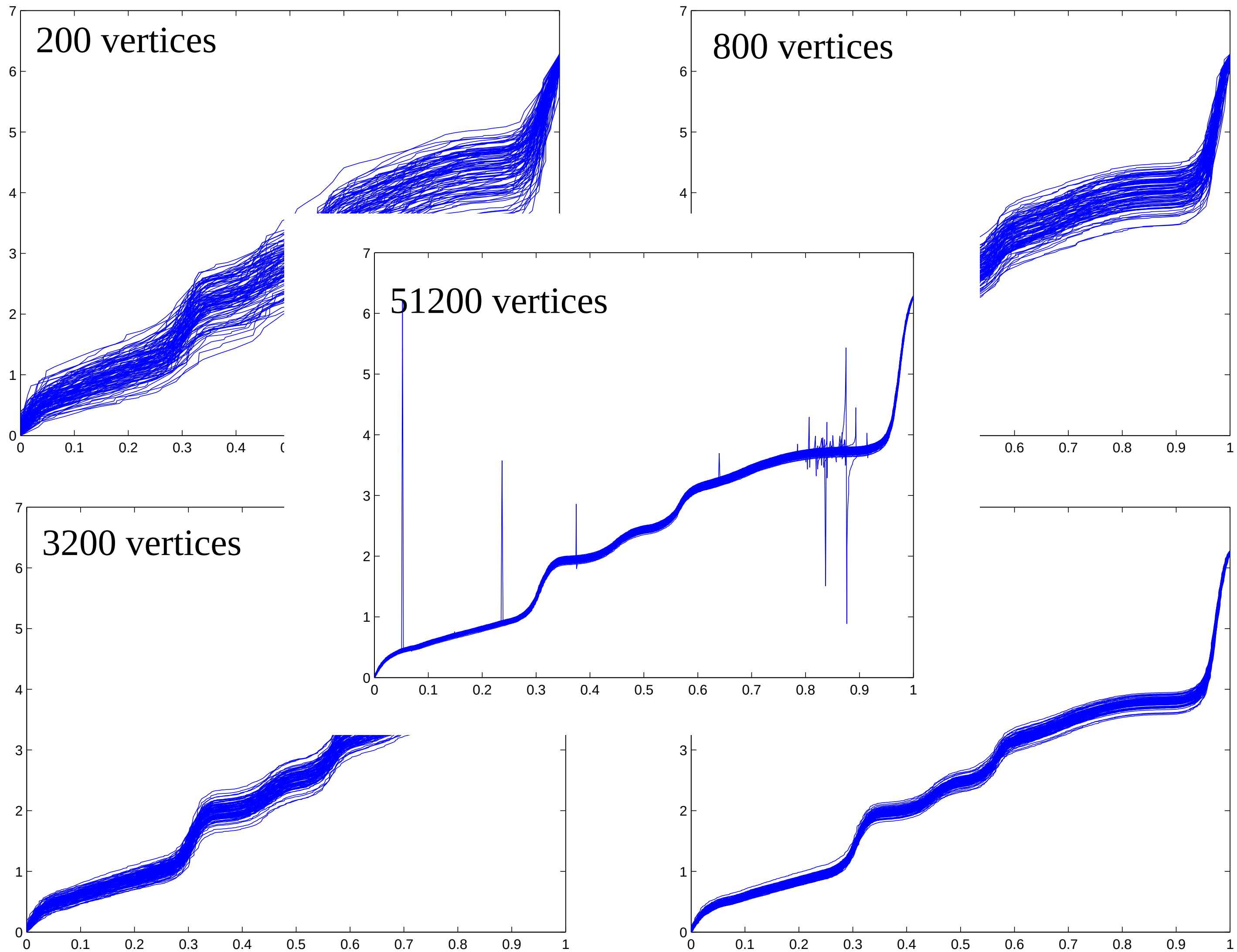
Harmonic Measure Trials



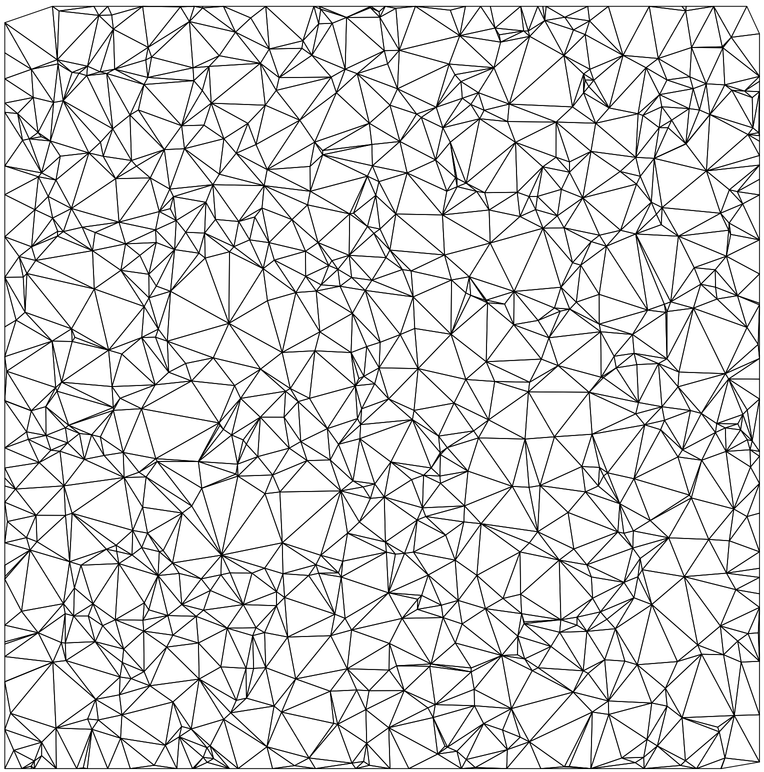
Harmonic Measure Trials



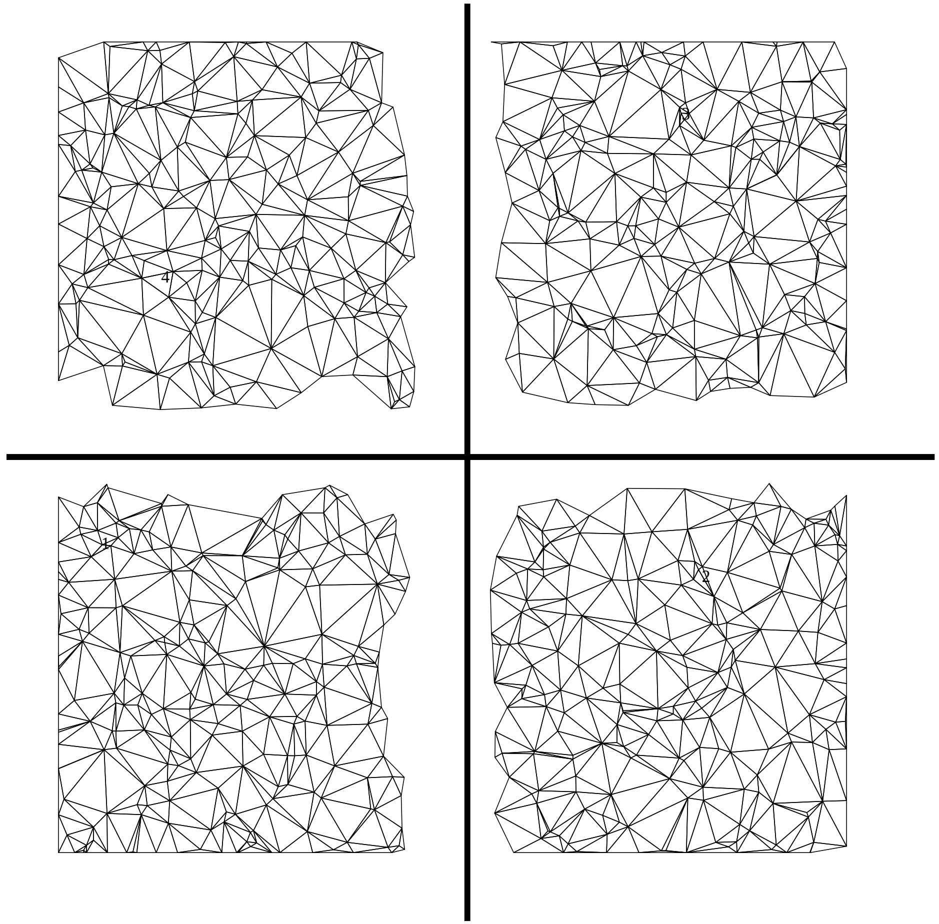
Harmonic Measure Trials



Intuition



800 points



$\sim$ 200 points each

4. What is a Random Triangulation?

- Subdivision Tilings
- Brain Mapping
- Random surfaces

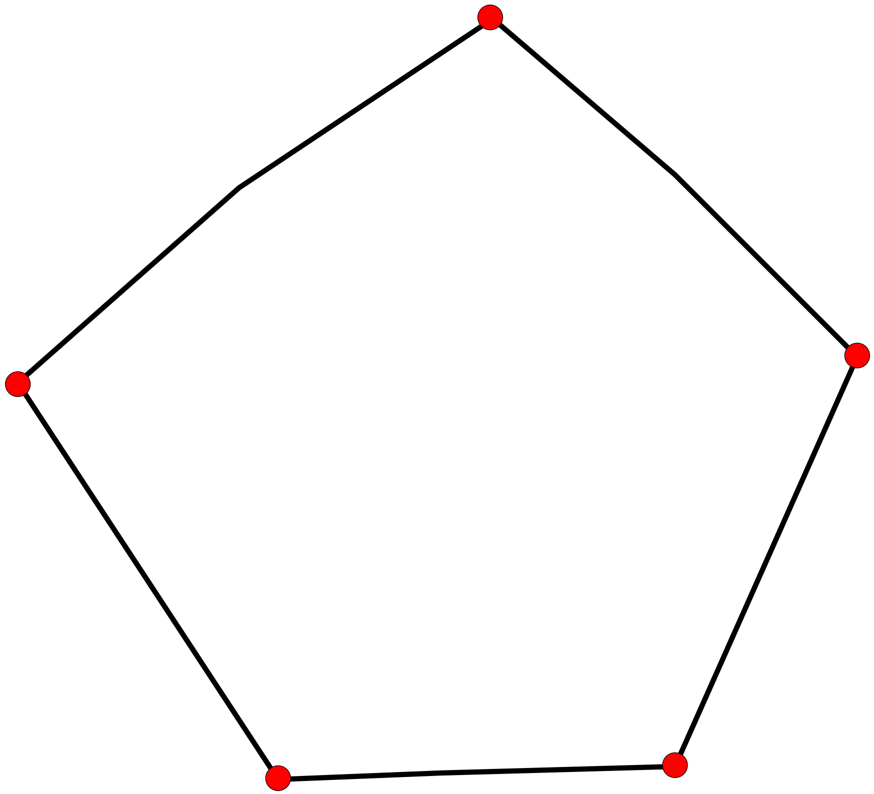
Subdivision Tilings

Studied by Jim Cannon, Bill Floyd, and Walter Parry in the context of Thurston's Geometrization Conjecture and Kleinian groups.

Subdivision Tilings

Studied by Jim Cannon, Bill Floyd, and Walter Parry in the context of Thurston's Geometrization Conjecture and Kleinian groups.

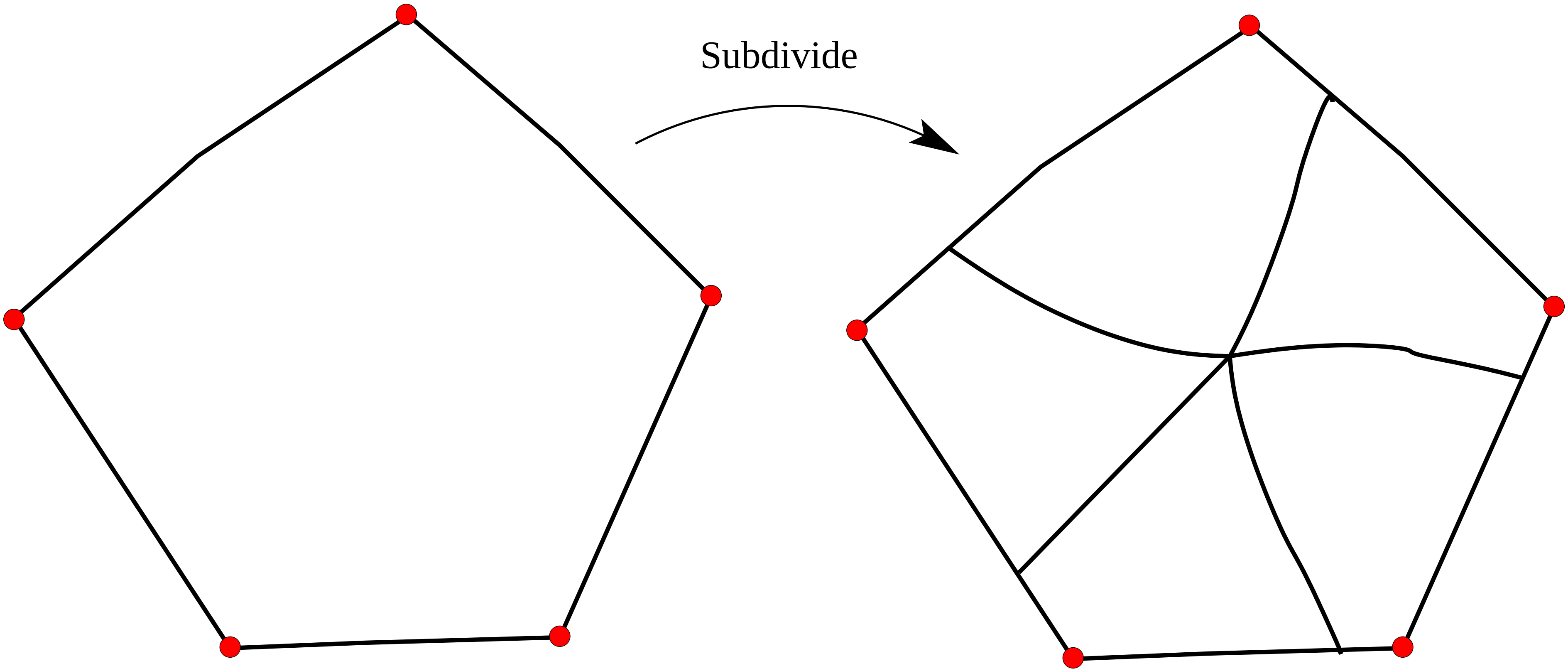
Their **Twisted Pentagonal** example goes like this:

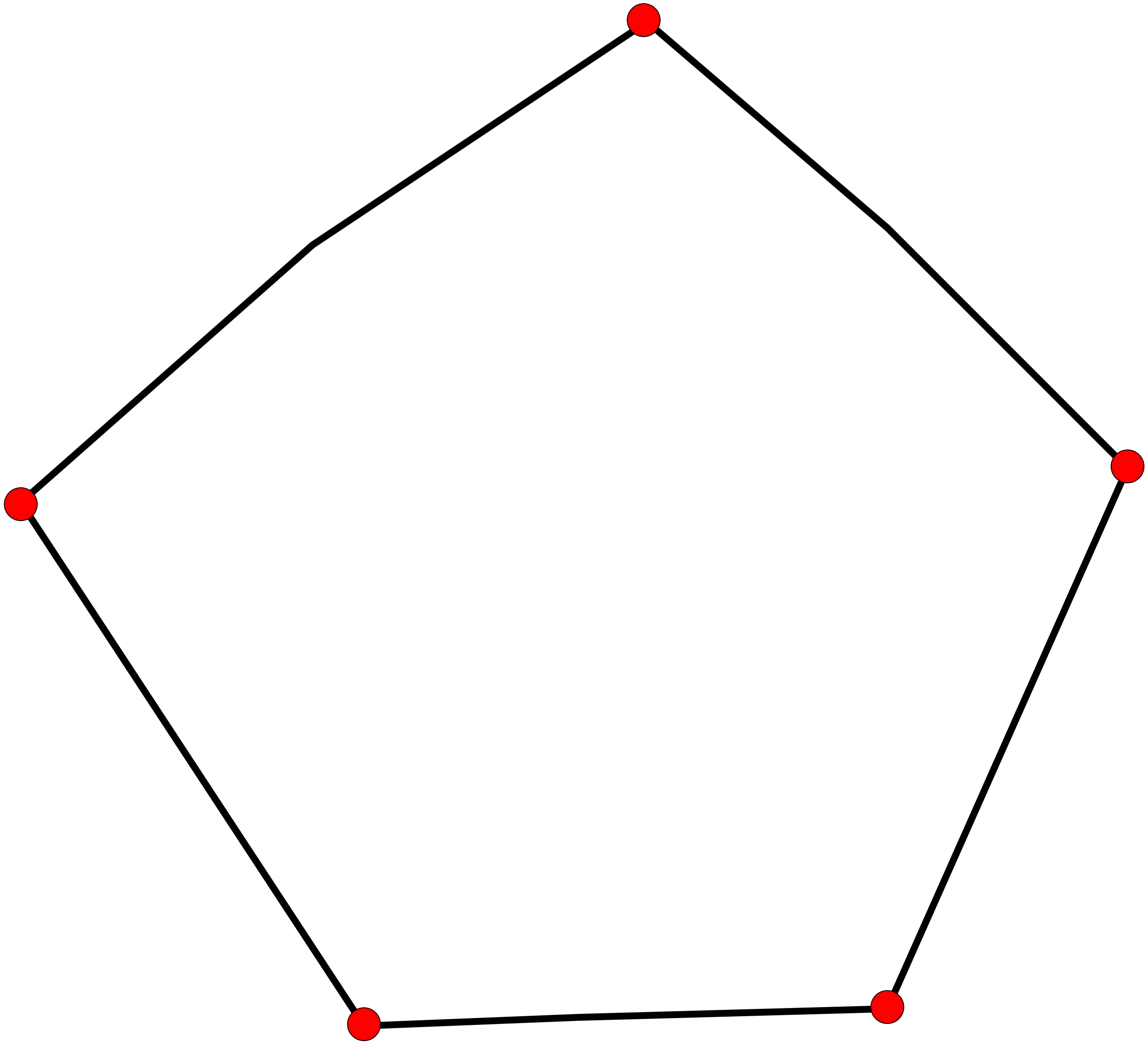


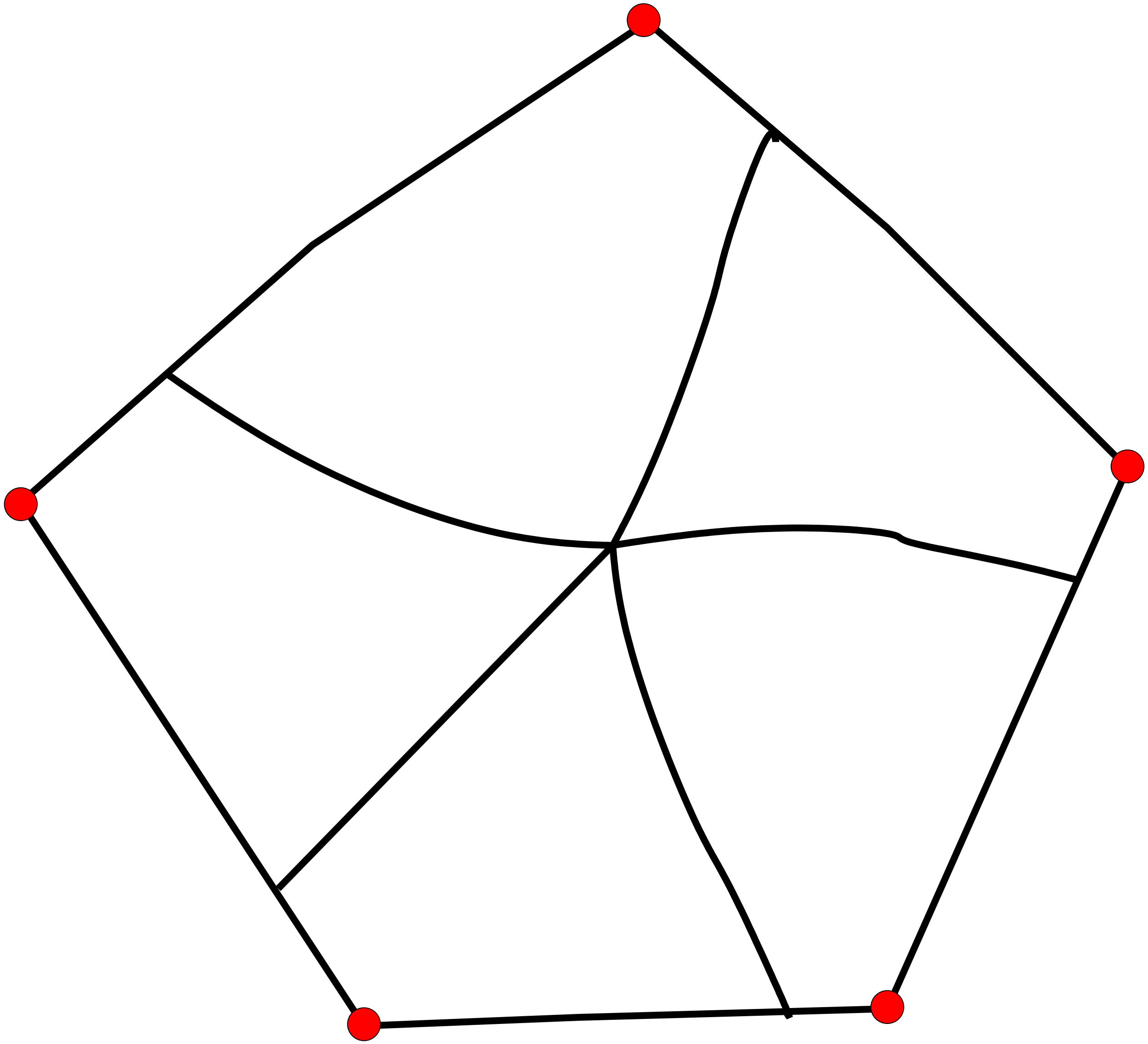
Subdivision Tilings

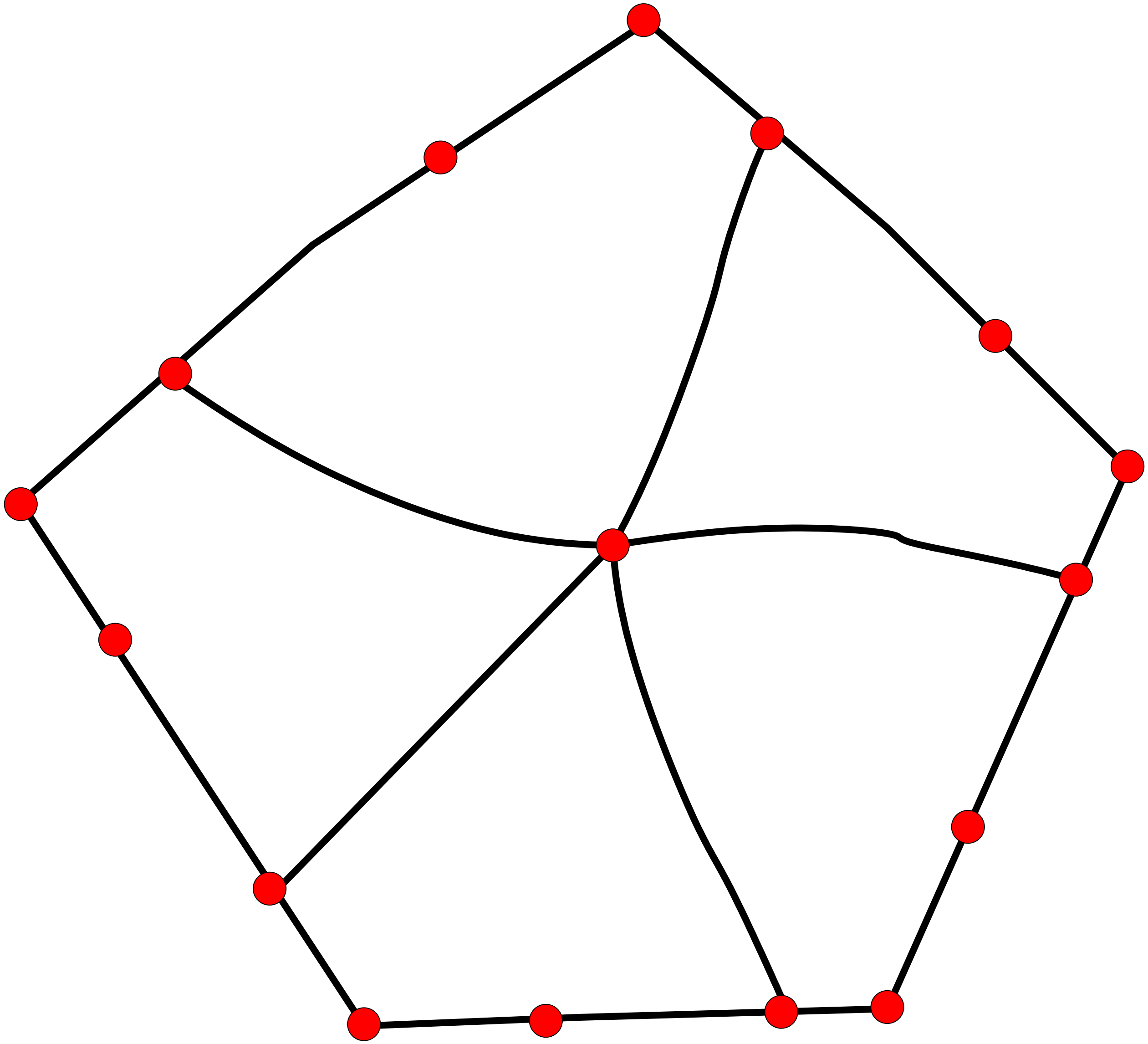
Studied by Jim Cannon, Bill Floyd, and Walter Parry in the context of Thurston's Geometrization Conjecture and Kleinian groups.

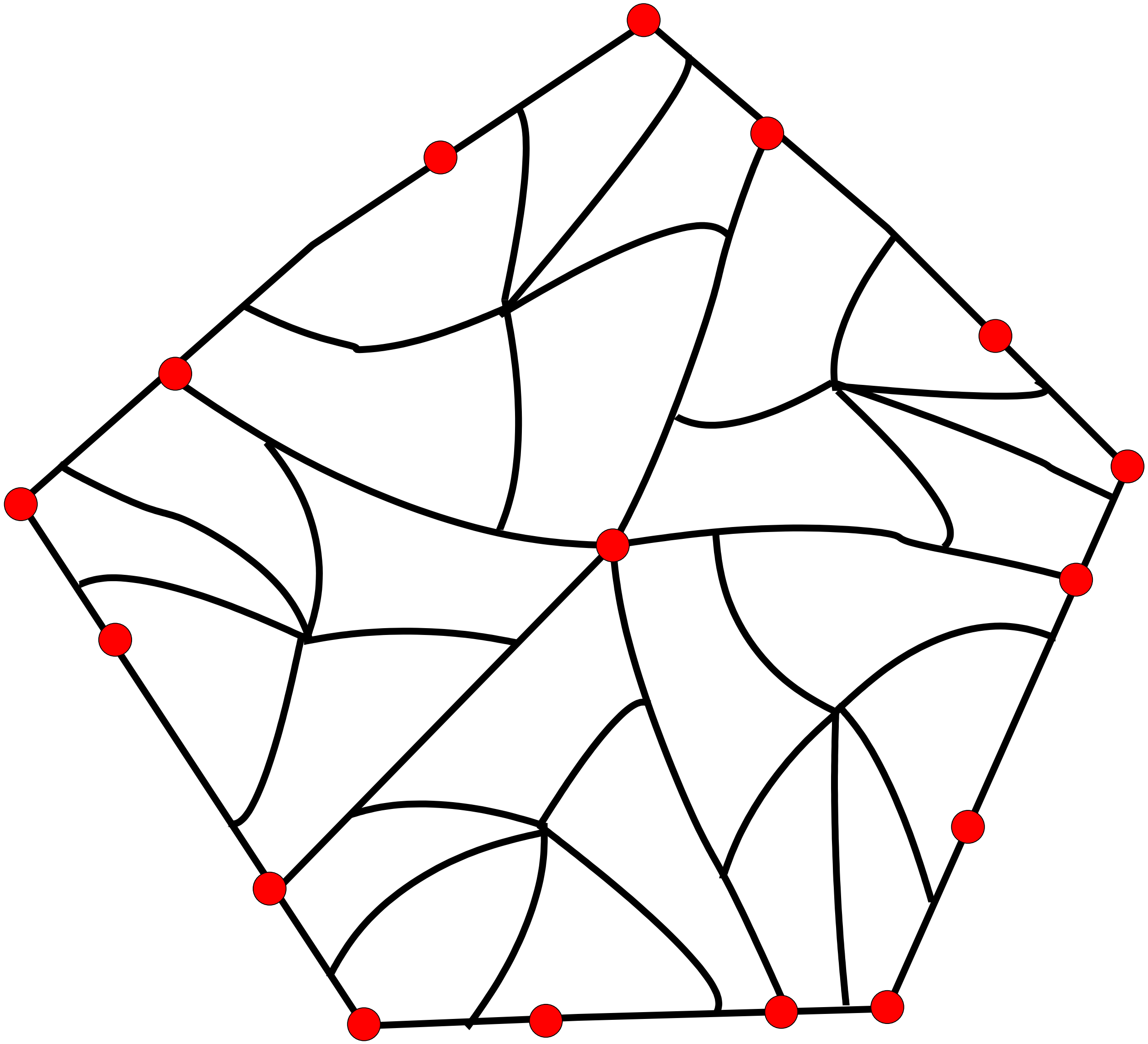
Their **Twisted Pentagonal** example goes like this:



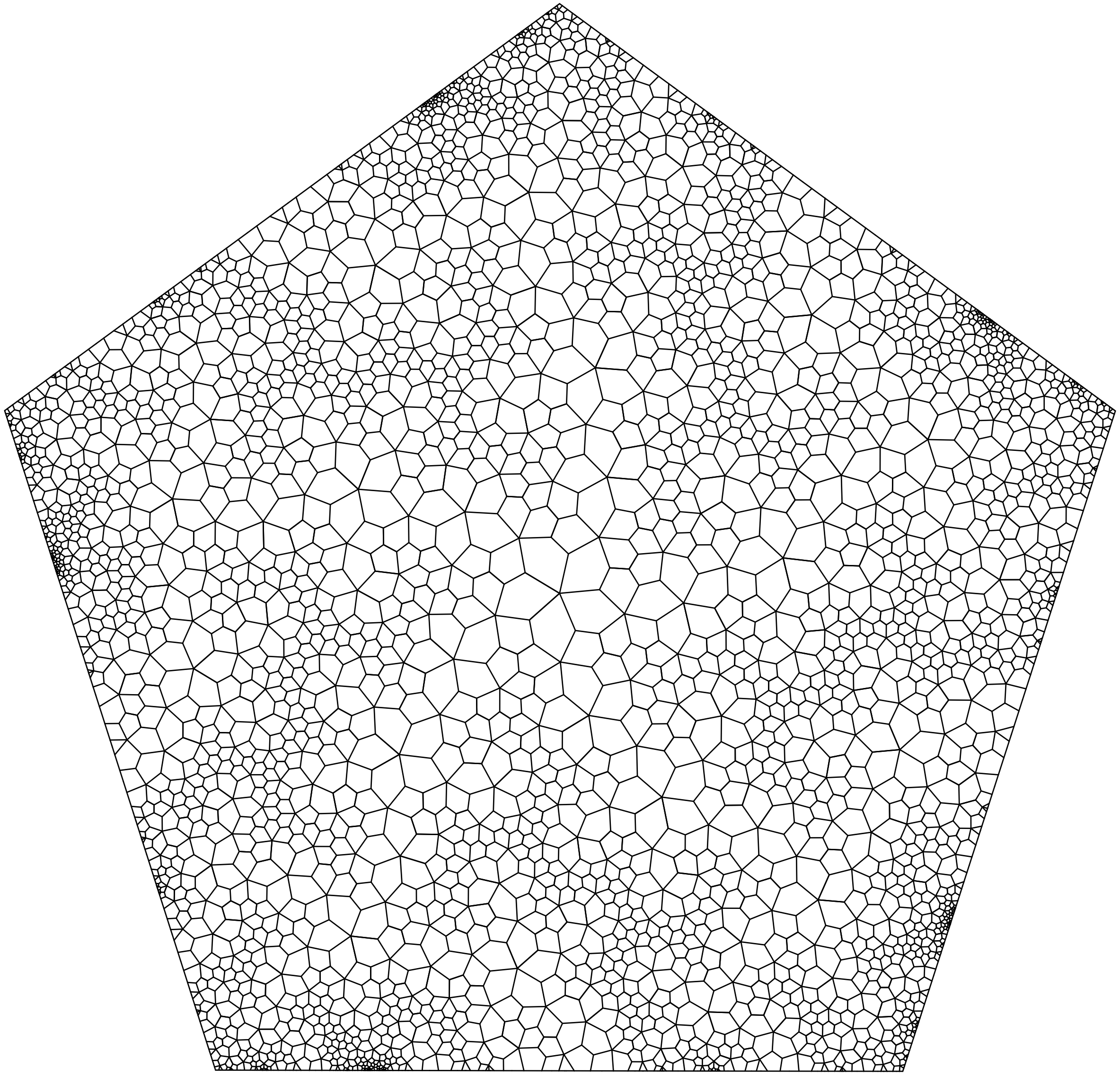






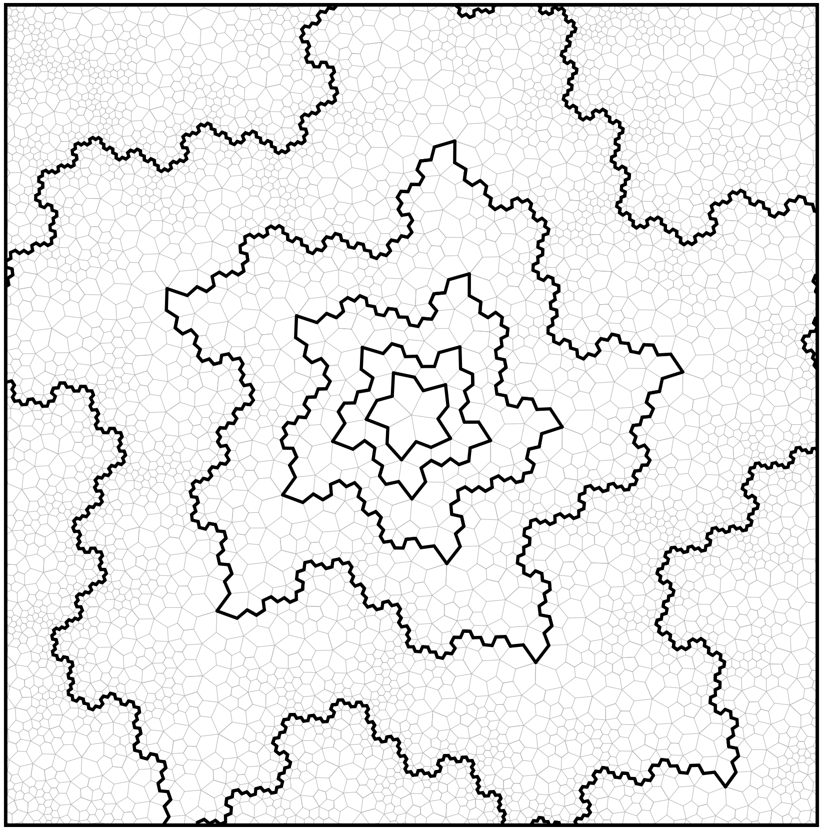


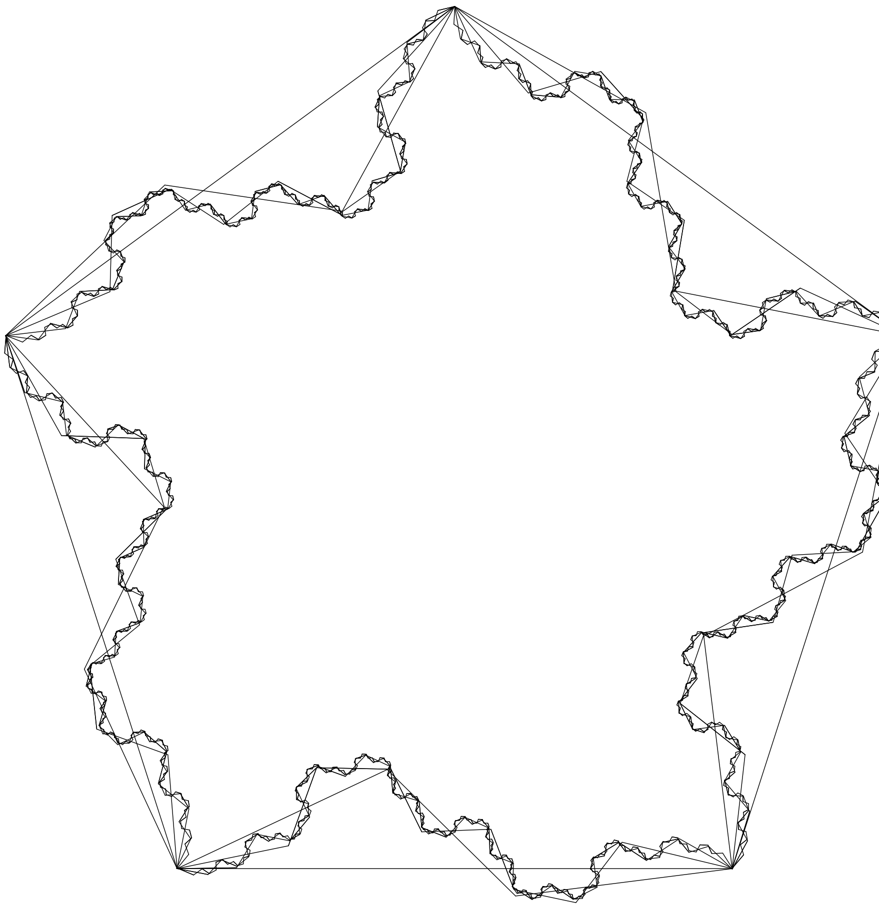
Circle Packed at Stage 7



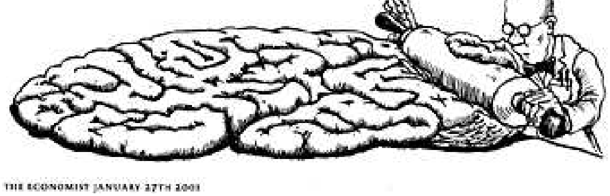
Unexpected Self-Similarity

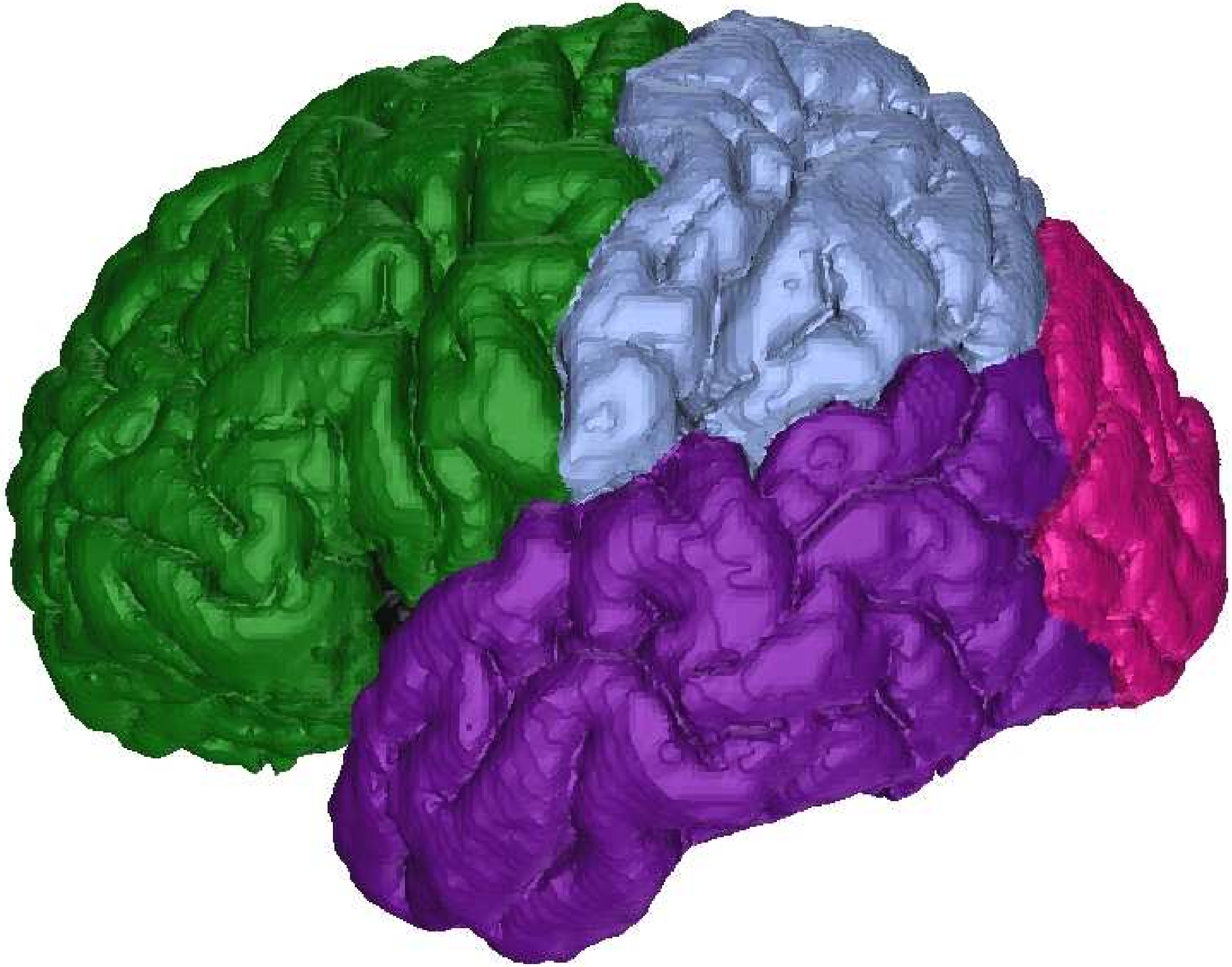
Unexpected Self-Similarity

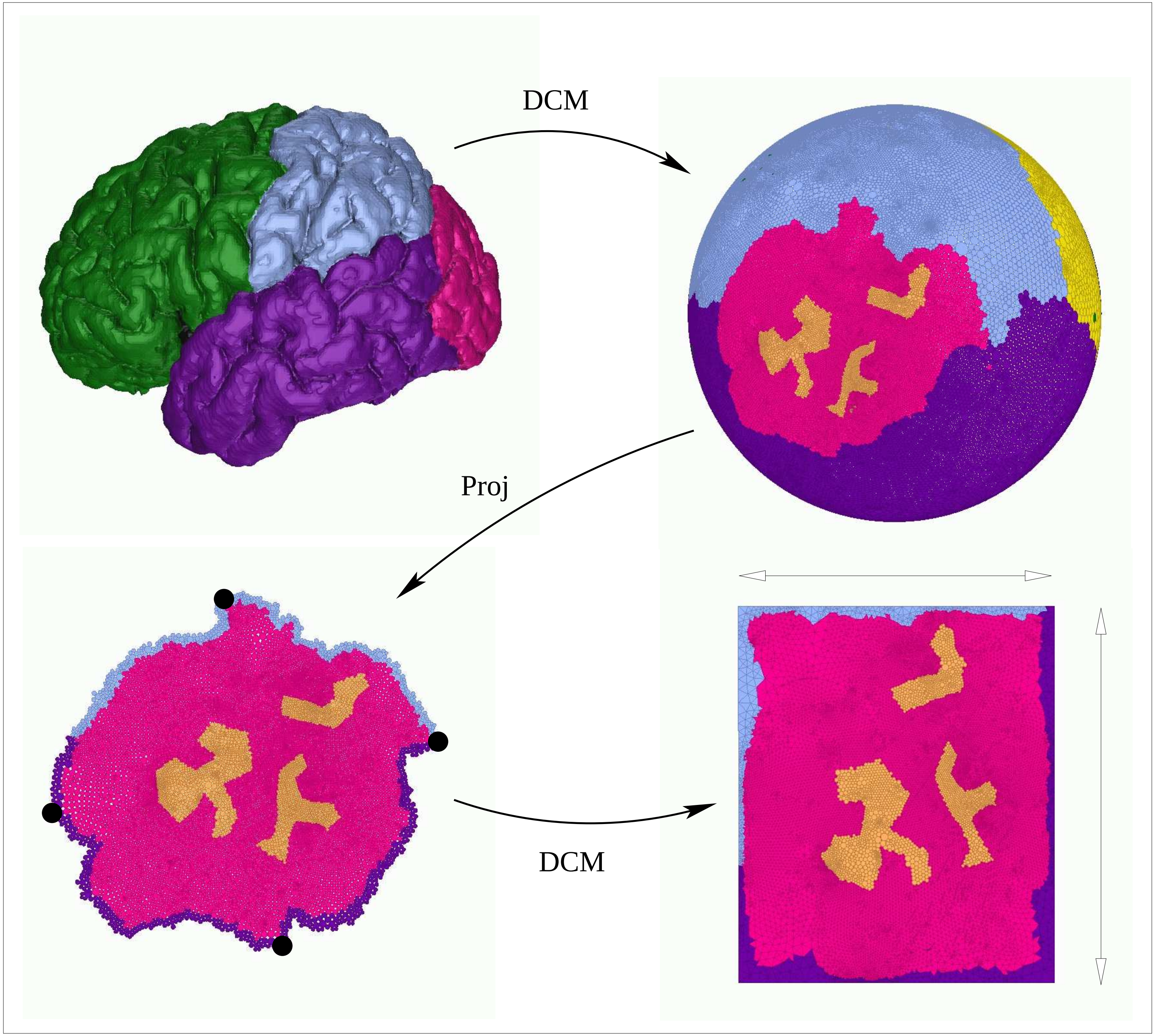




Brain Flattening

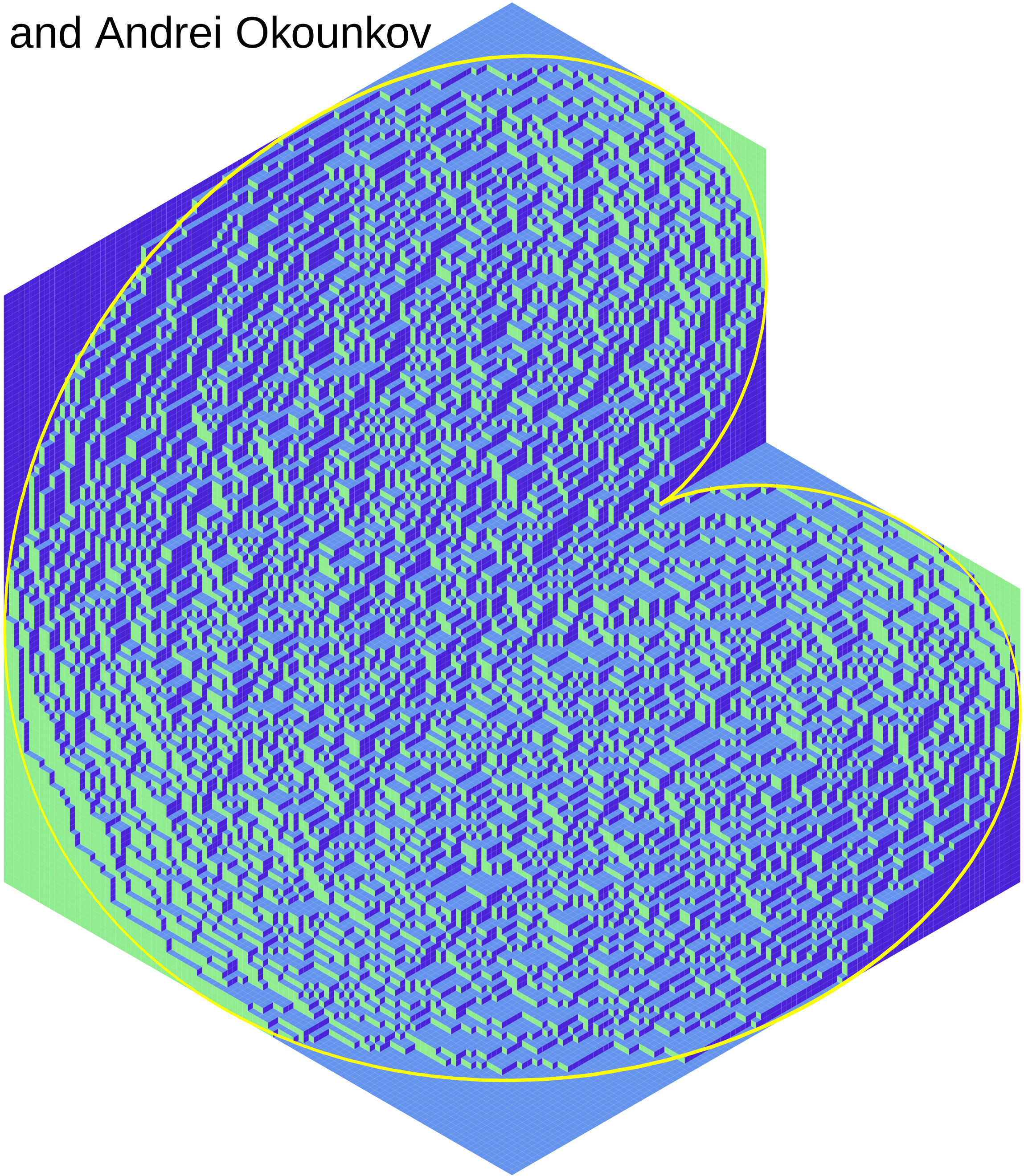





DCM
Proj
DCM

Random Surfaces in Physics

A “stepped surface”
Cf. Rick Kenyon and Andrei Okounkov



Acknowledgements

- Rick Kenyon for images (Kenyon/Okounkov)
- Jim Cannon, Bill Floyd, Walter Parry for tiling data
- Gerald Orick (Tennessee) for his new packing/layout algorithm
- Brain collaborators: Monica Hurdal, Phil Bowers, De Witt Sumners, David Rottenberg, Chuck Collins
- NSF for their support of this research