Schwarz-Christoffel mapping for difficult cases

Lehel Banjai
Institut für Mathematik, Zurich, Switzerland

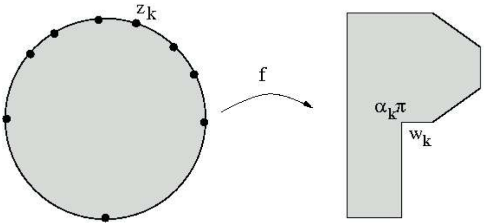
computational mathematics

Computational and Conformal Geometry April 20-22, SUNY Stony Brook

1. The crowding phenomenon
   - Conformal mapping of polygons
   - Stable computation of Schwarz-Christoffel integrals
   - The parameter problem
   - Numerical experiments
2. Thousands of vertices
   - Applying the Fast Multipole Method
   - A simple iteration for the parameter problem
   - Applications
3. Conclusions

Outline

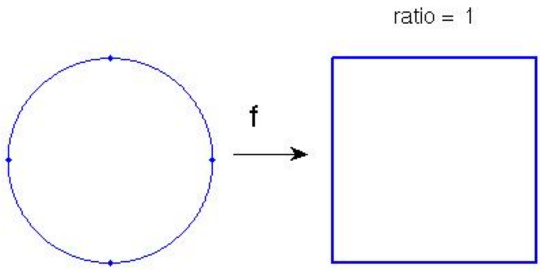
Schwarz-Christoffel formula



$$f(z) = A + C \int^z \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1} \mathrm{d}\zeta.$$

- w_k - **vertices**; $z_k = \exp(i\theta_k)$ - **prevertices**.
- Find $A, C, z_1, z_2, \ldots, z_k$ - **the parameter problem**.
- Three free real parameters - e.g. $z_1 = 1, z_{n-1} = -1, z_n = -i$.

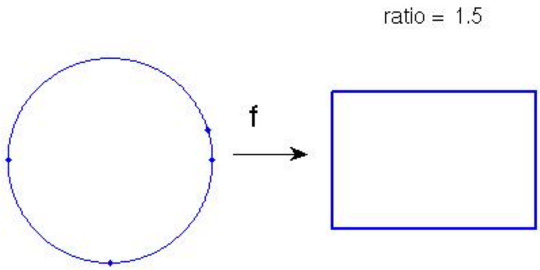
Conformal map to a rectangle



$$f(z) = C \int^{z} (\zeta - 1)^{-1/2} (\zeta - z_2)^{-1/2} (\zeta + 1)^{-1/2} (\zeta + i)^{-1/2} d\zeta + A.$$

$z_2 = 0 + i$

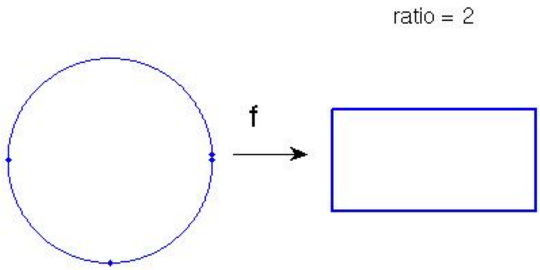
Conformal map to a rectangle



$$f(z) = C\int^{z}(\zeta-1)^{-1/2}(\zeta-z_2)^{-1/2}(\zeta+1)^{-1/2}(\zeta+i)^{-1/2}d\zeta + A.$$

$$z_2 = 0.953317517854098 + 0.301969717277250i$$

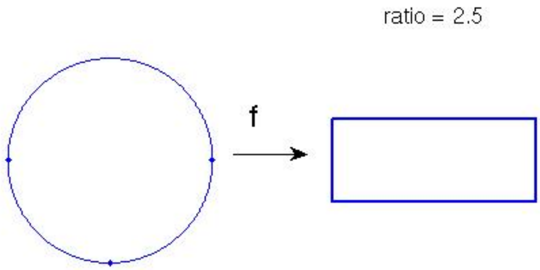
Conformal map to a rectangle



$$f(z) = C \int^{z} (\zeta - 1)^{-1/2} (\zeta - z_2)^{-1/2} (\zeta + 1)^{-1/2} (\zeta + i)^{-1/2} d\zeta + A.$$

$$z_2 = 0.998161862707872 + 0.060604420924140i$$

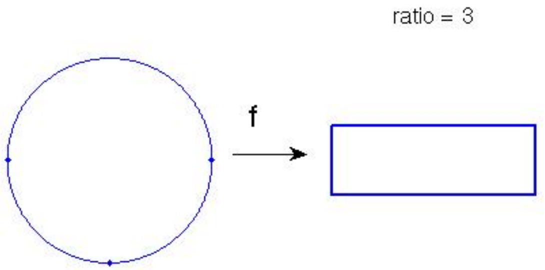
Conformal map to a rectangle



$$f(z) = C\int^z (\zeta-1)^{-1/2}(\zeta-z_2)^{-1/2}(\zeta+1)^{-1/2}(\zeta+i)^{-1/2}d\zeta + A.$$

$$z_2 = 0.999922362701654 + 0.012460680926077i$$

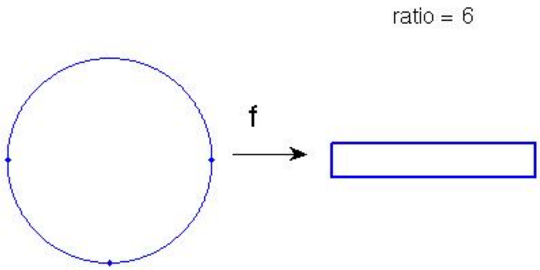
Conformal map to a rectangle



$$f(z) = C \int^{z} (\zeta - 1)^{-1/2} (\zeta - z_2)^{-1/2} (\zeta + 1)^{-1/2} (\zeta + i)^{-1/2} d\zeta + A.$$

$$z_2 = 0.999996661341993 + 0.002584048154957i$$

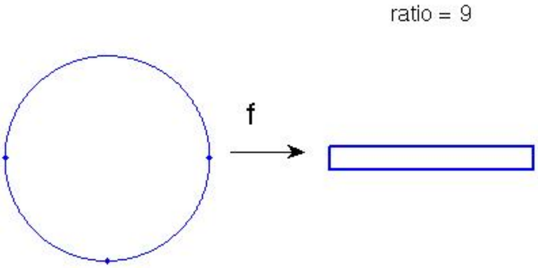
Conformal map to a rectangle



$$f(z) = C \int^z (\zeta - 1)^{-1/2} (\zeta - z_2)^{-1/2} (\zeta + 1)^{-1/2} (\zeta + i)^{-1/2} d\zeta + A.$$

$$z_2 = 0.999999999999978 + 0.000000208397196i$$

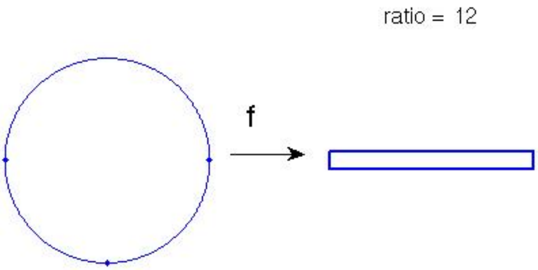
Conformal map to a rectangle



$$f(z) = C\int^{z}(\zeta-1)^{-1/2}(\zeta-z_2)^{-1/2}(\zeta+1)^{-1/2}(\zeta+i)^{-1/2}d\zeta + A.$$

$$z_2 = 1.000000000000000 + 0.000000000016818i$$

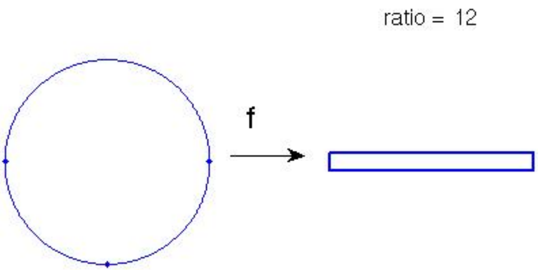
Conformal map to a rectangle



$$f(z) = C\int^z (\zeta-1)^{-1/2}(\zeta-z_2)^{-1/2}(\zeta+1)^{-1/2}(\zeta+i)^{-1/2}d\zeta + A.$$

$$z_2 = 1.000000000000000 + 0.000000000000001i$$

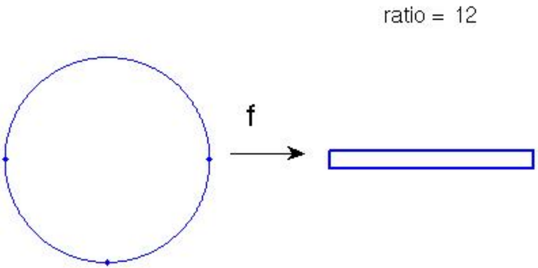
Conformal map to a rectangle



$$f(z) = C \int^{z} (\zeta - 1)^{-1/2} ((\zeta - 1) - (z_2 - 1))^{-1/2} (\zeta + 1)^{-1/2} (\zeta + i)^{-1/2} d\zeta + A.$$

$$\arg(z_2) - \arg(1) = 1.357168333126092 \times 10^{-15}$$

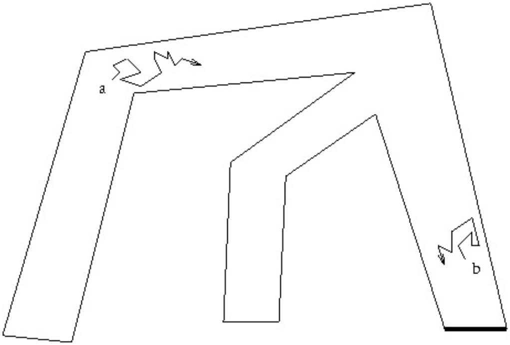
Conformal map to a rectangle



$$f(z) = C \int^z \prod_{k=1}^{4} ((\zeta - z_1) - (z_k - z_1))^{-1/2} d\zeta + A.$$

$$\arg(z_2) - \arg(1) = 1.357168333126092 \times 10^{-15}$$

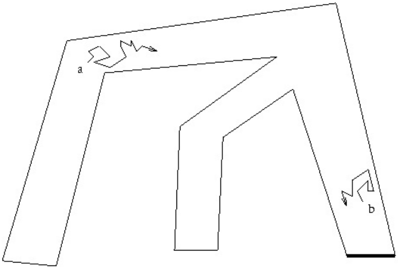
Spotting crowding in general domains



Let z_1, z_2 be mapped to the corners of the highlighted edge.

- For $f(0) = a$, $|z_1 - z_2| \ll 1$, and for $f(0) = b$, $|z_1 - z_2| \sim 1$, therefore region of crowding depends on the normalization.

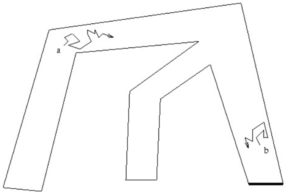
Spotting crowding in general domains



Let z_1, z_2 be mapped to the corners of the highlighted edge.

- For $f(0) = a$, $|z_1 - z_2| \ll 1$, and for $f(0) = b$, $|z_1 - z_2| \sim 1$, therefore region of crowding depends on the normalization.
- This observation used by the **C**ross **R**atio **D**elaunay **T**riangulation algorithm of Driscoll and Vavasis (1998).

Spotting crowding in general domains



Let z_1, z_2 be mapped to the corners of the highlighted edge.

- For $f(0) = a$, $|z_1 - z_2| \ll 1$, and for $f(0) = b$, $|z_1 - z_2| \sim 1$, therefore region of crowding depends on the normalization.
- This observation used by the **C**ross **R**atio **D**elaunay **T**riangulation algorithm of Driscoll and Vavasis (1998).
- Our plan: Consider each step of Trefethen's (1980) algorithm and modify for numerical stability
- The starting point: The SC Toolbox of Driscoll (1996,2005).

Floating point arithmetic

Theorem

If $x \in \mathbb{R}$ lies in the range of $\mathcal{R}$ then $\exists \delta \in \mathbb{R}$ s.t.

$$fl(x) = x(1+\delta), \quad |\delta| < u, \tag{1}$$

where u is the unit roundoff.

Model of arithmetic

$$fl(x \text{ op } y) = (x \text{ op } y)(1+\delta), \quad |\delta| \leq u,$$
$$\text{op} = +, -, *, /, \log, \exp, \sin, \cos, \ldots$$

where $x, y \in \mathcal{C} = \mathcal{R} + i\mathcal{R}$.

Computing the integrand accurately

Let

- z and z_k be such that $|z - z_k| \leq C_1 |z - z_j|$, $j = 1, 2, \ldots, N$,
- $\tilde{\eta} = (z - z_k)(1 + \delta_1)$, with $|\delta_1| \leq Cu$,
- $\tilde{\omega}_j = fl(z_j - z_k)$, and $\tilde{\beta}_j = fl(\beta_j)$.

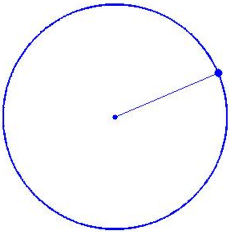
Then

$$fl\left(\prod_{j=1}^{N} (\tilde{\eta} - \tilde{\omega}_j)^{\tilde{\beta}_j}\right) = \prod_{j=1}^{N} (z - z_j)^{\beta_j}(1 + N\delta) + O(u^2),$$

where

$$|\delta| < (8 + 3C_1(C + 1) + 3\max_j \Big| \log |z - z_j| \Big|)u.$$

Quadrature and the "one-half rule"

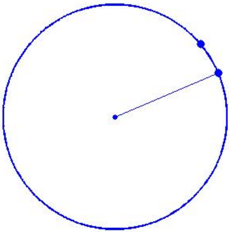


Gauss-Jacobi quadrature perfect for the singular integrals:

$$\int_0^{z_k} (\zeta - z_k)^{\beta_k} F(\zeta) d\zeta,$$

$F(\zeta)$ analytic in a neighbourhood of $[0, z_k]$.

Quadrature and the "one-half rule"



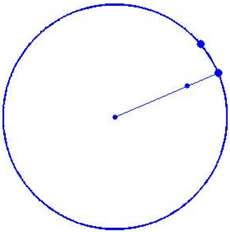
Gauss-Jacobi quadrature perfect for the singular integrals:

$$\int_0^{z_k} (\zeta - z_k)^{\beta_k} F(\zeta) d\zeta,$$

$F(\zeta)$ analytic in a neighbourhood of $[0, z_k]$.

A prevertex close to the integration path $\implies$ loss of accuracy.

Quadrature and the "one-half rule"



Gauss-Jacobi quadrature perfect for the singular integrals:

$$\int_0^{z_k} (\zeta - z_k)^{\beta_k} F(\zeta) d\zeta,$$

$F(\zeta)$ analytic in a neighbourhood of $[0, z_k]$.

A prevertex close to the integration path $\implies$ loss of accuracy.

The one-half rule (Trefethen 1980): No singularity may lie closer to an integration subinterval than one-half the length of that subinterval.

Implications of the one-half rule

Let $[a, b]$ be an integration interval. Then the one-half rule implies

$$\frac{1}{2} \leq \frac{|\zeta - z_j|}{|b-a|}, \quad \text{for all } \zeta \in [a,b] \text{ and } z_j \notin \{a, b\}.$$

Since no need to subdivide the interval unless a singularity is in the vicinity of the interval

$$\text{There exists } k, \text{ such that } \frac{|\zeta - z_k|}{|b-a|} \leq 1, \quad \text{for all } \zeta \in [a,b].$$

These assumptions now give us a bound on the constant C_1

$$|\zeta - z_k| \leq |b-a| \leq 2|\zeta - z_j|, \quad \text{for all } \zeta \in [a,b] \text{ and } z_j \notin \{a,b\}.$$

With these assumptions numerical stability of quadrature is proved.

Parameter problem

As primitive variables use *differences* of arguments of consecutive prevertices ($z_{N+1} := z_1$):

$$\phi_k = \arg z_{k+1} - \arg z_k, \quad k = 1, 2, \ldots, N.$$

The initially proposed normalization translates to

$$z_1 = 1, \quad \phi_{N-1} = \phi_N = \pi/2.$$

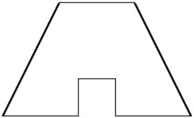
Parameter problem

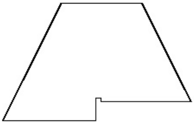
As primitive variables use *differences* of arguments of consecutive prevertices ($z_{N+1} := z_1$):

$$\phi_k = \arg z_{k+1} - \arg z_k, \quad k = 1, 2, \ldots, N.$$

The initially proposed normalization translates to

$$z_1 = 1, \quad \phi_{N-1} = \phi_N = \pi/2.$$





Sufficient and necessary condition:

$$\frac{\left|\int_{z_j}^{z_{j+1}} \prod_{k=1}^{n}(\zeta - z_k)^{\beta_k}\, d\zeta\right|}{\left|\int_{z_1}^{z_2} \prod_{k=1}^{n}(\zeta - z_k)^{\beta_k}\, d\zeta\right|} = \frac{|w_{j+1} - w_j|}{|w_2 - w_1|}, \quad j = 2, \ldots, n-2.$$

Sufficient and necessary condition:

$$\frac{\left|\int_{z_j}^{z_{j+1}} \prod_{k=1}^{n} (\zeta - z_k)^{\beta_k} d\zeta\right|}{\left|\int_{z_1}^{z_2} \prod_{k=1}^{n} (\zeta - z_k)^{\beta_k} d\zeta\right|} = \frac{|w_{j+1} - w_j|}{|w_2 - w_1|}, \quad j = 2, \ldots, n-2.$$

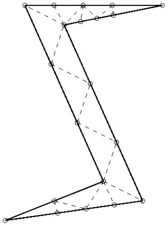
Change of variables to obtain an unconstrained system:

$$\psi_k = \log\left(\frac{\phi_{k+1}}{\phi_1}\right), \quad k = 1, \ldots, n-3.$$

The variables ϕ_k can be recovered by the formulas

$$\phi_1 = \frac{\pi}{\sum_{k=1}^{n-3} e^{\psi_k} + 1}, \; \phi_k = e^{\psi_{k-1}} \phi_1, \; k = 2, \ldots, n-2.$$

The CRDT (Driscoll, Vavasis 1998)



(*a*) Find $N-3$ quadrilaterals Q_i with vertices $w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}$ so that a conformal map is well-conditioned near the four vertices

(*b*) For this to be possible vertices may have to be added

(*c*) As primitive variables use the *the cross ratios*:

$$\rho(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4}) = \frac{(w_{i_4} - w_{i_1})(w_{i_2} - w_{i_3})}{(w_{i_3} - w_{i_4})(w_{i_1} - w_{i_2})}$$

A uniformly good initial guess

Use cross ratios for the initial guess:

$$\rho(\tilde{z}_{j_1}, \tilde{z}_{j_2}, \tilde{z}_{j_3}, \tilde{z}_{j_4}) = -|\rho(w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4})|$$

The guess is uniformly good (Bishop 2003)

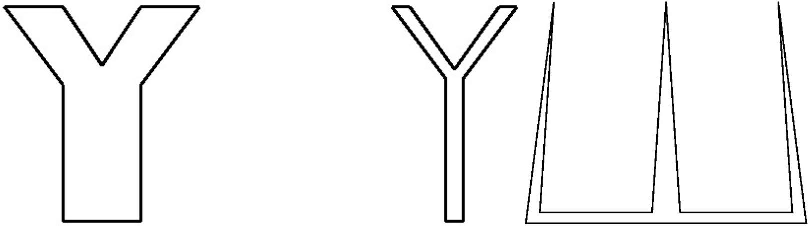
There exists $K < \infty$ (independent of the polygon) s.t.

$$d_{QC}(\{\tilde{z}_k\}, \{z_k\}) \leq \log K;$$

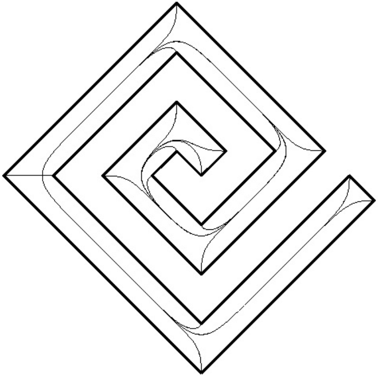
where

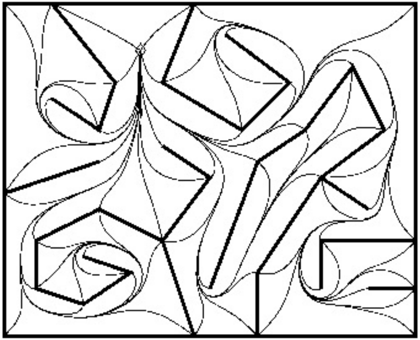
$$d_{QC}(\tilde{\mathbf{z}}, \mathbf{z}) = \inf\{\log K : \exists K-\text{quasiconf. } h : D \to D \text{ s.t. } h(\mathbf{z}) = \tilde{\mathbf{z}}\}.$$

A gallery of polygons I



A gallery of polygons II





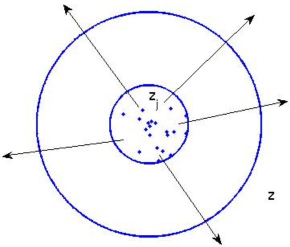
Results

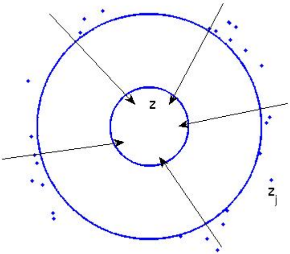
polygon	CRDT	SCTS	SCTS with initial guess
Y (crowded)	30/7.4s	164/9.1s	34/1.8s
Fork	109/102.3s	181/6.1s	91/5.2s
Spiral	71/33.7s	185/14s	41/3.5s
Emma's maze	33/59.1s	1326/411.3s	368/114.0s
Rectangle (200)	31/161.0s	28/2.8s	3/2.4s
Rectangle (250)	28/154.8s	–/–	–/–
∞ Y	–/–	97/7.1s	–/–

Number of evaluations of the nonlinear function are shown against the time in seconds needed to compute a map to the listed polygons.

Outline

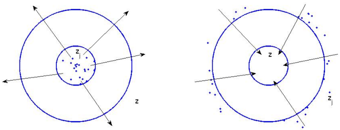
Multipole and local expansions





Consider $G(z) = \sum_{j=1}^{L} \beta_{k_j} \log(z - z_{k_j})$.

Multipole and local expansions



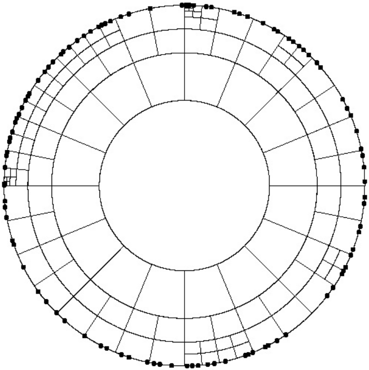
Consider $G(z) = \sum_{j=1}^{L} \beta_{k_j} \log(z - z_{k_j})$. To speed up computations use *multipole expansions*:

$$G(z) \approx a_0 \log(z - z_0) + \sum_{m=1}^{p} \frac{a_m}{(z - z_0)^m}, \text{ for } |z_j| \leq r \text{ and } |z| > R$$

and *local expansions*:

$$G(z) \approx \sum_{m=0}^{p} b_m (z - z_0)^m, \quad \text{for } |z| \leq r \text{ and } |z_j| > R; \text{ error } \leq C \left(\frac{r}{R}\right)^{p+1}.$$

Fast evaluation of SC integral



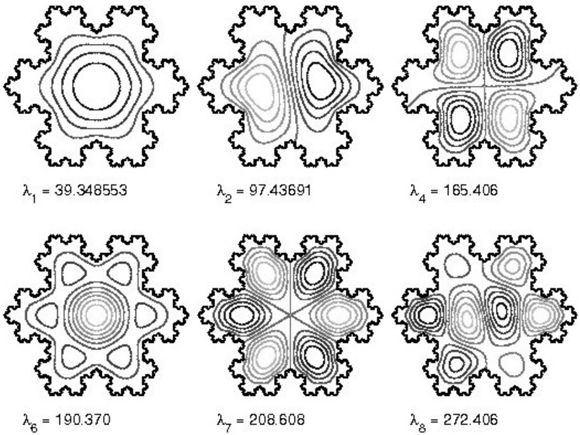
- The same numerical stability considerations apply
- Can compute $f(z)$ at any N points in $O(N \log N)$ time
- But how to find the prevertices!?

Davis's iteration (1979)

$$\phi_j^{(n+1)} := k\phi_j^{(n)} \frac{|w_{j+1} - w_j|}{|w_{j+1}^{(n)} - w_j^{(n)}|}, \quad j = 0, 1, \ldots, N-1.$$

- k - is chosen so that $\sum \phi_j = 2\pi$
- Works best if three *non-consecutive* prevertices are fixed and three Davis's iterations performed
- Known to diverge (even locally) for some polygons
- When it works, $O(N \log N)$ cost to find the prevertices

Eigenfrequencies of the Koch snowflake



Outline

1. The crowding phenomenon
   - Conformal mapping of polygons
   - Stable computation of Schwarz-Christoffel integrals
   - The parameter problem
   - Numerical experiments
2. Thousands of vertices
   - Applying the Fast Multipole Method
   - A simple iteration for the parameter problem
   - Applications
3. Conclusions

Conclusion

- Standard methods for the computation of SC maps can be adapted to perform robustly in the presence of extreme crowding.
- Main ingredients of the modified method:
  - Care for numerical stability at each stage
  - CRDT initial guess uniformly close to the solution
- Question: What is the best way (cheap, stable, effective) to obtain an initial guess? Other possibilities include: Zipper (see talk of Marshall), see talk of Bishop.
- Related questions: understanding of Davis's algorithm, alternative efficient iteration for $N \gg 1$