MAT331 Project 2 – Nonescaping sets

April 6, 2023

In class we have recently looked at the behavior of escaping sets for iterations of complex polynomials.

Let A_∞ be the escaping set for a polynomial p, it is the set of points z which escape to infinity under iteration of p.

The *Julia set* $J(p)$ of $p(z)$ is the boundary of A_∞.

In this project you will investigate some alternate ways to draw the Julia set of a polynomial. The necessary mathematical facts are summarized below.

- Let z_0 be any point in $J(p)$. Fix $N > 0$. Then

  $$J(p) \approx J_N(p) := \{w : p^{\circ n}(w) = z_0 \text{ for some nonnegative integer } n \leq N\} \tag{0.1}$$

  This means that once we have a single point in $J(p)$, we can approximate the rest of $J(p)$ by solving the equation $f^{\circ n}(w) = z_0$.

- If p is a polynomial and $p(z) = z$, and $|p'(z)| \geq 1$, then $z \in J(p)$. The point z is called a *fixed point* because $p(z) = z$, and it is *non-attractive* because near z, p is approximated by an expanding (or neutral) map.

- If $p(z) = z^2 + c$, it always has a non-attractive fixed point.

1 Programming Tasks

1. Find a non-attractive fixed point for $p(z) = z^2 + 1/4$. Then write a python function that takes an input c (complex number) and returns a non-attractive fixed point for $p(z) = z^2 + c$.

2. Using the non-attractive fixed point you found in the previous part, write a function that lists the points in $J_N(p)$, defined above. The input should be c (a complex number) and N (the number of steps in the approximation) and the output should be a list of complex numbers.

3. Plot this list (for $c = \{1/4, i, -0.12 + 0.74i\}$ and a reasonable choice of N) using pyplot, like in the solutions to HW5.

4. Now we'll draw the points in a different way. Use imshow instead of pyplot to plot the points that you found above. Hint: you could loop through the grid and set it to be black iff the little square contains one of the points from the list. A better way would be to go through the list and then figure out which square it lands inside, and then set the value of the appropriate entry in the matrix. (Why is it better?)

5. Make one of the modifications/generalizations of your code from the following list below:

   (a) Make your code work with any polynomial f, not just those of the form $z^2 + c$. You will need the following mathematical fact: If p is a polynomial, there is some integer n such that $f^{\circ n}$ has a nonattractive fixed point z_0. Furthermore, (0.1) with this z_0 yields an approximation to the Julia set of p.

   (b) The method is a bit wasteful in the sense that some points are hit too often, while other points are not hit enough. Improve your code by keeping track (using the matrix) of how many times each square has been hit.

   Choose an M and stop taking pre-images of points that have more than M hits. This will allow you to use a larger N. You'll need to experiment with M to get good results. Compare this with the original method in terms of time spent vs quality of picture.

   (c) Instead of taking both pre-images of a point, corresponding to the positive and negative square root in the quadratic formula, try taking only one solution, at random (so either positive sqrt or negative sqrt with equal probability).

   Compare this with the original method in terms of time spent vs quality of picture.

2 Written Report

The report should include

- A comparison of the algorithms you implemented in 2,4, and 5, along with the original method shown in lecture. Discuss the qualitative differences in the pictures generated, and the differences in efficiency/run time.
- A detailed written description and explanation of the modification/generalization you made in Q5.
- Any unexpected programming or math challenges you encountered in this assignment.

You will submit the following on blackboard. Include the following:

- A .ipynb notebook file named `code_firstname_lastname.ipynb` with all your code from the Programming tasks section. Make sure to clearly separate the various parts. Also, test this notebook by restarting it and running from the beginning; this should produce no errors.
- A pdf file named `report_firstname_lastname.pdf` with your work for questions 7 and 8. The exact format of this report is up to you. It can be hand-written (legibly) and then scanned (legibly) into a pdf document. Or it can be be made on a computer. It must be in pdf form (it is generally easy to convert other forms such as .doc to pdf).