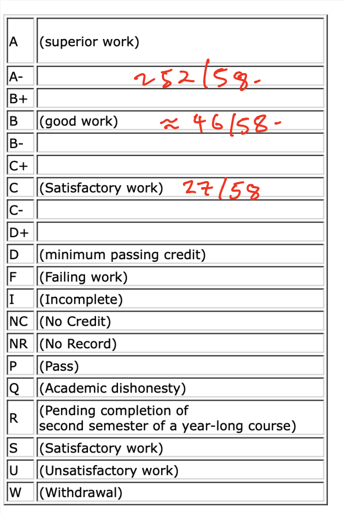
# MAT303: Calc IV with applications

Lecture 9 - March 8 2021

# Today

- About the midterm, and the grading criteria
- Euler's method (Ch 2.4)
- Second order linear differential equations (Ch 3.1)
    - Where they arise
    - Homogeneous equations
    - Linear independence
    - Principle of superposition



| | |
|---|---|
| A | (superior work) |
| A- | 252/58- |
| B+ | |
| B | (good work) ≈ 46/58- |
| B- | |
| C+ | |
| C | (Satisfactory work) 27/58 |
| C- | |
| D+ | |
| D | (minimum passing credit) |
| F | (Failing work) |
| I | (Incomplete) |
| NC | (No Credit) |
| NR | (No Record) |
| P | (Pass) |
| Q | (Academic dishonesty) |
| R | (Pending completion of second semester of a year-long course) |
| S | (Satisfactory work) |
| U | (Unsatisfactory work) |
| W | (Withdrawal) |

1. Integrating Factor
2. Slope Field
3. Separable
4. Exact Eqn.
5. Salt Water application
6. Equilibri - Bifurcation.

# Review: Tangent line approximation

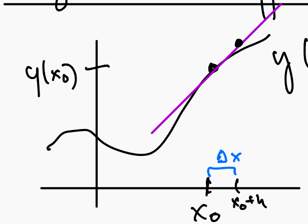
## Motivation:

What can we say about soln to

$$\frac{dy}{dt} = f(t,y)$$

when we can't solve it analytically.

## Tangent line approximation



$$y(x_0+h) \approx y(x_0) + \overbrace{\underset{\smile}{h}\,\underset{\smile}{y'(x_0)}}^{\Delta y}$$

$$\Delta y \approx \Delta x \cdot \frac{dy}{dx}$$

## Euler Method:

Iterating the tangent line approx. gives Euler's method.

**Example:** $\frac{dy}{dt} = \overset{2(1)+5}{2t+y}, \quad y(1) = 5.$

What is $y(2)$?

| $t$ | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|
| $y$ | 5 | 6.4 | | | | |

We know $y(1) = 5$.

Now find approx. value at $y = 1.2$

$$y(1.2) \approx y(1) + 0.2\,\underset{\frac{dy}{dt}}{(7)}$$

$$= 5 + 1.4 = 6.4$$

# Numerical solutions to DEs: Euler's method

**Euler Method:**

Iterating the tangent line approx. gives Euler's method.

**Example:** $\frac{dy}{dt} = \overset{2(1)+5}{2t + y}$, $y(1) = 5$.

Actual soln is

$y = 9e^{t-1} - 2(t+1)$

so

$y(2) = 9e - 4$

$\approx 18.4645....$

what is $y(2)$?

| $t$ | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|
| $y$ | 5 | 6.4 | 8.16 | 10.352 | 13.0624 | 16.34 |

We know $y(1) = 5$.

Now find approx. value at $y = 1.2$

$y(1.2) \approx y(1) + 0.2\,(7)$ $\frac{dy}{dt}$

$= 5 + 1.4 = 6.4$

Now let's find $y(1.4)$ approximately.

$y(1.4) \approx y(1.2) + 0.2\,(2t + y)$ $\frac{dy}{dt}$

$+\,0.2\,(2(1.2) + 6.4)$

$= 6.4 + 0.2 \times 8.8$

$= 8.16$

Can continue this way until we get

$y(2) \approx 16.84.$

| $h$ | Approx value. |
|---|---|
| 0.2 | 16.34 |
| 0.1 | 17.3437 |
| 0.01 | 18.3433. |
| 0.001 | 18.45... |
| 0.0001 | 18.46... |
| Exact soln: | |

The error in the approximation is $O(h)$.

Doubling the steps, increases the error goes down by a factor of $\frac{1}{2}$

* Many other numerical method exists e.g. Runge-Kutta. It's $O(h^2)$, much more accurate.

* Helps us understand what a DE is.

# Ch3: Second order linear differential equations

# What is a second order linear differential equation?

First order linear DEs:

$$r(x)y' + p(x)y = q(x)$$

Second order Linear DEs.

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

# Constant coefficients: a concrete example

Second order linear DEs.

$p(x)y'' + q(x)y' + r(x)y = g(x)$

Constant coeff.

$ay'' + by' + cy = g(x)$

Constant coeff, homogeneous.

$ay'' + by' + cy = 0$

Example: Find solutions to

$$y'' + 5y' + 6y = 0 \qquad (1)$$

"Guess" $y = e^{rt}$, $r$ is a parameter.

Substitute into (1):

$y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$$\Rightarrow r\,e^{rt} + 5re^{rt} + 6e^{rt} = 0$$

$$\Rightarrow (r^2 + 5r + 6)e^{rt} = 0$$

$$\Rightarrow (r^2 + 5r + 6) = 0$$

$$\Rightarrow (r+2)(r+3) = 0$$

$\Rightarrow r = -2$ or $r = -3$.

So solns are

$y = e^{-2t}$ and $y = e^{-3t}$.

Are these the only solutions?

No, $y = Ae^{-2t}$ and $y = Ae^{-3t}$ both work.

$$y'' + 5y' + 6y = A4e^{-2t} + A(-2)5\,e^{-2t} + A6e^{-2t} = 0$$

Any more?

$y = Ae^{-2t} + Be^{-3t}$.

$$y' = A(-2)e^{-2t} + B(-3)e^{-3t}$$

$$y'' = A4e^{-2t} + B9e^{-3t}$$

$$y'' + 5y' + 6y = \underbrace{A4e^{-2t} + 5A(-2)e^{-2t} + 6Ae^{-2t}}_{=0} + \underbrace{B9e^{-3t} + 5B(-3)e^{-3t} + 6Be^{-3t}}_{=0} = 0.$$

The space of solns is a vector space spanned by $y = e^{-2t}$ and $y = e^{-3t}$.

# Principle of $\underbrace{\text{superposition}}_{\text{linear combination.}}$ for homogeneous equations

if $y_1$ and $y_2$ are solns to a $\underbrace{\text{homogeneous}}_{(g=0).}$ linear DE:

$$p(t)y'' + q(t)y' + r(t)y = 0 \quad (1)$$

Then for any $C_1, C_2,$

$$C_1y_1 + C_2y_2$$

is a soln to (1)

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Explanation:

Plug $C_1y_1 + C_2y_2$ into (1):

$$p(t)(C_1y_1'' + C_2y_2'')$$
$$+ q(t)(C_1y_1' + C_2y_2')$$
$$+ r(t)(C_1y_1 + C_2y_2)$$

$$= p(t)C_1y_1'' + p(t)C_2y_2''$$
$$+ q(t)C_1y_1' + q(t)C_2y_2'$$
$$+ r(t)C_1y_1 \quad r(t) + C_2y_2$$

$$= = C_1\left(p(t)y_1'' + + q(t)y_1' + + r(t)\cdot y_1\right) + C_2\left(p(t)y_2'' + q(t)y_2' + r(t) + y_2\right)$$

$$= C_1 \cdot 0 + C_2 \cdot 0$$

$$= 0.$$

# Principle of superposition for homogeneous equations

# Recall: Complex numbers and Euler's identity

Let $i$ be the "number" satisfying $i^2 = -1$.

Euler's identity.

$$e^{ix} = \cos x + i \sin x$$

# Constant coefficients: another example

Find the general soln for

$$y''+2y'+2y=0.$$

$y(t)$

$t=\pi$

Try guessing $y=e^{rt}$

$y'=re^{rt}$

$y''=r^2e^{rt}$

So our constraint is.

$$r^2e^{rt}+2re^{rt}+2e^{rt}=0$$

$$(r^2+2r+2)e^{rt}=0.$$

$$(r^2+2r+2)=0$$

$$r=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2.}=\frac{-2\pm 2i}{2}$$

$$(2i)(2i)=4i^2=4(-1)=-4$$

$$=-1\pm i$$

So $y=e^{(-1+i)t}$ $y=e^{(-1-i)t}$ are "solutions".

Euler's Identity.

$$e^{ix}=\cos x+i\sin x$$

$i^2=-1.$

$y_1(t)$

$y_1(\pi)$ real.

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So $y_1=e^{-t+it}=e^{-t}e^{it}=e^{-t}(\cos t+i\sin t)$

and $y_2=e^{-t-it}=e^{-t}e^{-t(i)}=e^{-t}(\cos(-t)+i\sin(-t))$

$$=e^{-t}(\cos t-i\sin t)$$

are solutions.

But they're complex

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But $y_1+y_2$ is a soln,

$y_1+y_2=2e^{-t}\cos t$. is a soln.

# Constant coefficients: the general case

# Summary

Today:

- Euler's method for numerically solving DEs
- Second order linear DEs
  - What are they
  - The constant coefficient case
    - Sub in $y = e^{rt}$ to find basis solutions
    - Combine basis solutions to get general solutions
  - Principle of superposition

Next time: continue Ch 3.1,
general second order linear DE

# Existence and uniqueness

# Linear independence of solutions

# Wronskian