MAT303: Calc IV with applications

Lecture 8 - March 1 2021

Recently:

Ch 2.1 population models

$$\frac{dP}{dt} = (\beta - \delta)P$$

Ch 2.2 Analysis of $\dfrac{dP}{dt} = f(P)$

- Equlibria
- Stability
- Bifurcation

HW4 P22.

Today:

Ch 2.3 Acceleration/Velocity Models

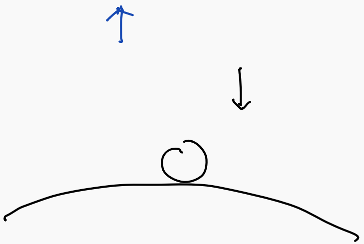
$$\frac{dv}{dt} = F$$

Question:

- How do assumptions about air resistance affect the terminal velocity?

Question:

- Suppose a lunar lander is falling towards the moon.
- We want the velocity at landing to be 0.
- When should we turn on the upwards thrusters?



Question:

- How fast do we have to launch from the ground to escape the earth's orbit?

Newton's second law:

$$\frac{dv}{dt} = F$$

Today we will only consider:

$$\frac{dv}{dt} = \text{External Force} + \text{Gravity} + \text{Resistance}$$

Question:

- How do assumptions about air resistance affect the terminal velocity of a falling object?

Question:

- Suppose a lunar lander is falling towards the moon.
- We want the velocity at landing to be 0.
- When should we turn on the upwards thrusters?

Question:

- How fast do we have to launch from the ground to escape the earth's orbit?

acceleration

Newton's second law:

$$m\frac{dv}{dt} = F$$

Today we will only consider:

$$m\frac{dv}{dt} = \text{External Force} + \text{Gravity} + \text{Resistance}$$

mass of earth

Physical inputs:

- $v = \dfrac{dx}{dt}$ - velocity is the rate of change of position
- Gravitational force = $\dfrac{Gm_1m_2}{r^2}$
    - If close to earth, assume $Gm_1/r^2 \approx g$ constant.
- Air resistance: proportional to v or v^2

  (in opposite direction to motion)

Question:

- How do assumptions about air resistance affect the terminal velocity of a falling object?

Question:

- Suppose a lunar lander is falling towards the moon.
- We want the velocity at landing to be 0.
- When should we turn on the upwards thrusters?

Question:

- How fast do we have to launch from the ground to escape the earth's orbit?

$m\frac{dv}{dt}$ = External Force + Gravity + Resistance

$g > 0.$

Suppose that
- There is no external force
- Gravity is constant
- Resistance is proportional to v

$$m\frac{dv}{dt} = -gm - kv$$

$$\frac{dv}{dt} = -g - \frac{k}{m}v$$

$\rho > 0$

- Can solve the separable equation for $v(t)$
  - Can find y(t)
- Can also analyze using techniques from Ch2.2
- Main qualitative phenomenon: terminal velocity is mg/k .

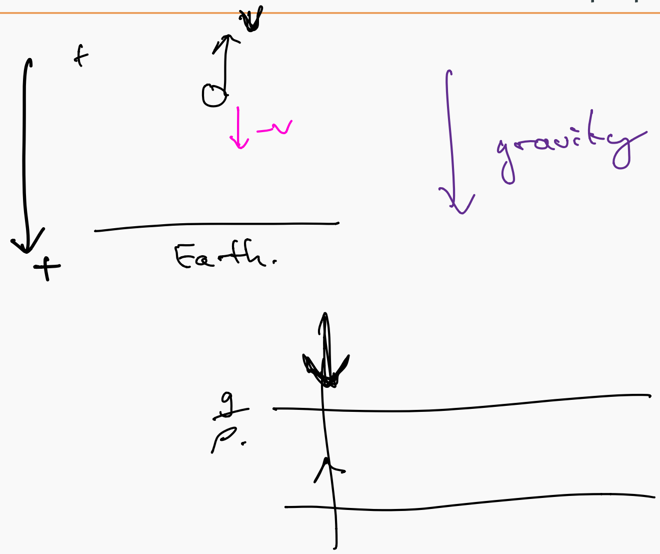
Sep. variables.

$$\frac{1}{-g-\rho v}\,dv = dt$$

Velocity.

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}.$$

As $t \to \infty$, $v \to -\frac{g}{\rho}$. terminal velocity.



Position:

$$y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}). \qquad (9)$$

$m\frac{dv}{dt}$ = External Force + Gravity + Resistance

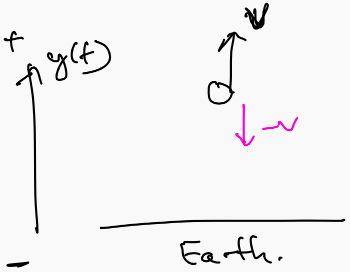
Suppose that

- There is no external force
- Gravity is constant
- Resistance is proportional to v^2

$$m\frac{dv}{dt} = -gm - \underbrace{kv|v|}$$

* if $v>0$, $kv|v| = kv^2$

* if $v<0$, $kv|v| = kv(-v) = -kv^2$



Careful about the signs!

- Can solve the separable equation for $v(t)$
  - Can find y(t)
- Can also analyze using techniques from Ch2.2
  - Main qualitative phenomenon: terminal velocity is $\sqrt{mg/k}$.

Our eqn: $\frac{dv}{dt} = -g - \rho v|v|$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

if $v>0$, $\frac{dv}{dt} = -g - \rho v^2$

Soln is.

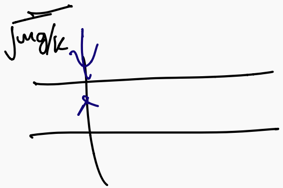
$$v(t) = \sqrt{\frac{g}{\rho}}\tan\left(C_1 - t\sqrt{\rho g}\right) \quad \text{with} \quad C_1 = \tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right).$$

if $v<0$

$$\frac{dv}{dt} = -g - \rho v(-v) = -g + v^2\rho.$$

$$\Downarrow$$

$$v(t) = \sqrt{\frac{g}{\rho}}\tanh\left(C_2 - t\sqrt{\rho g}\right) \quad \text{with} \quad C_2 = \tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right).$$



$m\frac{dv}{dt}$ = External Force + Gravity + Resistance

Suppose that

- External force is 0
- Gravity = $\frac{GM}{r^2}$
- Resistance 0

$$m\frac{dv}{dt} = -\frac{GM}{r^2}m$$

$$\frac{dv}{dt} = -\frac{GM}{r^2}$$

$$v = \frac{dr}{dt}$$

So $\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v.$

So $\frac{dv}{dr}v = -\frac{GM}{r^2}$

So $v\,dv = -\frac{GM}{r^2}dr$

Question:

• How fast do we have to launch from the ground to escape the earth's orbit?

So

$$\frac{1}{2}v^2 = \frac{GM}{r} + C$$

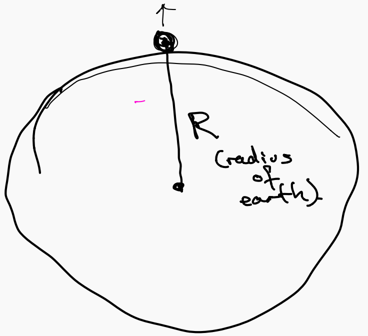
$$\frac{1}{2}v(0)^2 = \frac{GM}{R} + C$$

So $C = \frac{1}{2}v(0)^2 - \frac{GM}{R}.$

So

$$\frac{1}{2}v^2 = \frac{GM}{r} + \frac{1}{2}v(0)^2 - \frac{GM}{R}.$$

$$v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) + v(0)^2.$$



How big does $v(0)$ have to be to escape orbit?

As $r \to \infty$,

$$v^2 = -\frac{2GM}{R} + v(0)^2$$

want to be positive otherwise we turn around and crash into earth.

So we want $v(0)^2 > \frac{2GM}{R}.$

i.e. $v(0) \geq \sqrt{\frac{2GM}{R}}. \approx 6.95$ mi/s.

$m\frac{dv}{dt}$ = External Force + Gravity + Resistance

Suppose that
- External force is 0 or $E > 0$
- Gravity = $\frac{GM}{r^2}$
- Resistance 0

$$m\frac{dv}{dt} = E - \frac{GM}{r^2}m.$$

Set up differential equations ~~before and after thrust~~ is activated:

$$\frac{dv}{dt} = \cancel{T} - \frac{GM}{r^2}$$

0 or 4

$$v = \frac{dr}{dt}, \quad \text{so} \quad \frac{dv}{dt} = \frac{dv}{dr}v$$

So

$$v\frac{dv}{dr} = T - \frac{GM}{r^2}.$$

So

$$\frac{1}{2}v^2 = Tr + \frac{GM}{r} + C$$

Question:

- Suppose a lunar lander is falling towards the moon.
- We want the velocity upon landing to be 0.
- When should we turn on the upwards thrusters? Assume that they provide constant thrust.

At what height.

Assume:
- Initial altitude is y(0)=53km
- Initial velocity is v(0)=1477km/h downwards.
- Thrusters give $T = 4m/s^2$ deceleration
- Mass of moon is $M = 7.35 \cdot 10^{22}$ kg of moon.
- Radius is $R = 1740$ km

We want velocity at landing to be 0.

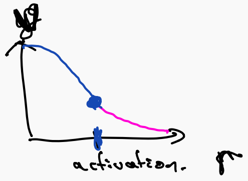
$$0 = 4R + \frac{GM}{R} + C_1$$

Initial alt. is 53km, initial vel is 1477

$$\frac{1}{2}v(0)^2 = \frac{GM}{(y(0)+R)^2} + C_2$$

$$\frac{1}{2}v^2 = 4r + \frac{GM}{r} + C_1$$

$$\frac{1}{2}v^2 = \frac{GM}{r} + C_2$$



So, at time of activation,

$$4r + \frac{GM}{r} + C_1 = \frac{GM}{r} + C_2$$

Consider:

$$\frac{dx}{dt} = \boxed{x(4-x) - h}$$

How does the qualitative nature of the solutions change when we modify h?

https://www.desmos.com/calculator/u3x5w62sde

One interpretation: How does the number of equilibrium points depend on h?

A bifurcation point for a parameter is a point at which the number of equilibrium solution changes.

when $h=4$, there is one equilibrium.

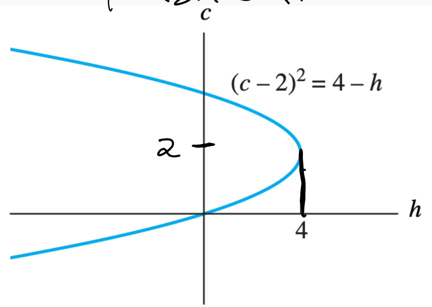


FIGURE 2.2.12. The parabola $(c-2)^2 = 4-h$ is the bifurcation diagram for the differential equation $x' = x(4-x) - h$.

Bifurcation diagram is a plot of parameter vs. the location of the equilibrium points.

Because $x(4-x)-4=0$ has one solution!

$$4x - x^2 - 4 = 0$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0.$$

Consider:

P21

$$\frac{dx}{dt} = kx - x^3$$

Draw the bifurcation diagram.

Find Equilibria:

$\frac{dx}{dt} = 0$ so $kx - x^3 = 0$

so $x(k - x^2) = 0$.

if $k > 0$:

$c = 0$, $c = \sqrt{k}$, $c = -\sqrt{k}$

if $k = 0$

$c = 0$.

if $k < 0$

$c = 0$