MAT303: Calc IV with applications

Lecture 7 - February 24 2021

Midterm 1:

- Next Wednesday in lecture, proctored over zoom, upload to gradescope.
- Allowed: textbook, lecture notes
- You will also be given a table of useful integrals.
- A random selection of students will be asked to set up a 10 minute meeting with me in the week after the exam to discuss their solutions
  - It is only to verify that you did not cheat
  - It's not meant to be very intense, it's usually pretty easy to determine between
    - Someone who cheated
    - Someone who did not

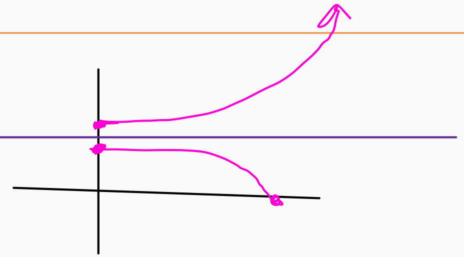
11 - 1.6

2.1

2.2.

Last time: Ch 2.1 population models

$$\frac{dP}{dt} = (\beta - \delta)P$$



We saw how changing β, δ affected the behavior of the solution.

Today:

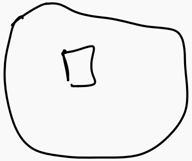
Ch 2.2 Analysis of $\underbrace{\frac{dP}{dt} = f(P)}$ as opposed to $\frac{dP}{dt} = f(P, t)$

First order autonomous.

Example: Let k be a constant. What can we say about the solution to

$$\frac{dx}{dt} = -k(x - A)$$

$x(t) =$ temperature at time t.



Solution:

$$x(t) = A + \underbrace{(x_0 - A)e^{-kt}}_{\to 0 \text{ as } t \to \infty}$$

Conclusion:

- As $t \to \infty$, temperature x approaches A.
- Also, $x = A$ is a solution.

Actually, can find equilibrium solution just by setting $\dfrac{dx}{dt} = 0$:

$$\frac{dx}{dt} = 0 \implies -k(x - A) = 0$$

$$\implies x = A.$$

"not changing wrt time" = "$\frac{dx}{dt} = 0$".

Whenever

$$\frac{dx}{dt} = f(x),$$

the equilibrium solutions are the zeros of $f(x)$.

Example: Let k be a constant. What are the equilibrium solutions?

$$\frac{dx}{dt} = kx(M - x)$$

Logistic Equation

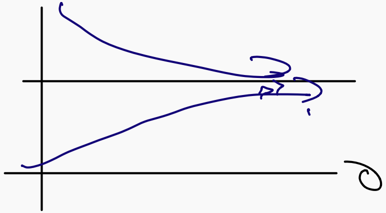
Set $\frac{dx}{dt} = 0$

$\Rightarrow kx(M-x) = 0$

$\Rightarrow x = 0$ or $x = M$.

Question: As $t \to \infty$, what is the long run behavior of the system?

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$



Question: As $t \to \infty$, which equilibrium solution do we approach?

Definition: An equilibrium solution $x = c$ to $\dfrac{dx}{dt} = f(x)$

is **stable** if:

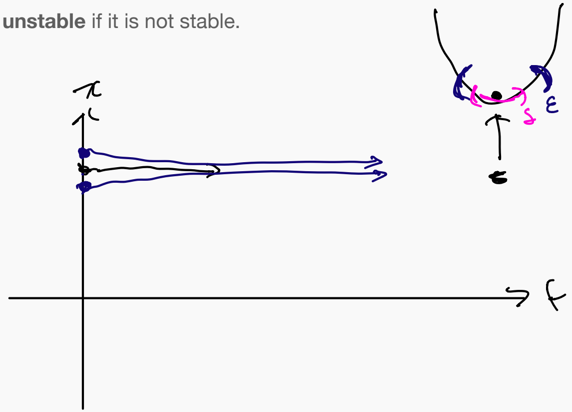
Solutions x(t) starting near c end up staying near c in the long run.

For all $\varepsilon>0$, there exists $\delta>0$.

if $|x(0)-c|<\delta$ then $|x(t)-c|<\varepsilon$.

Definition: An equilibrium solution $x = c$ to $\dfrac{dx}{dt} = f(x)$

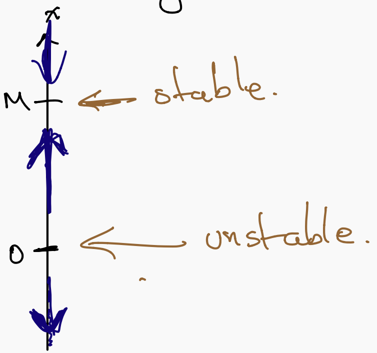
is **unstable** if it is not stable.



Example: Let k be a constant. Equilibrium solutions are x=M and x=0.

Are they stable?

Phase diagram. $\dfrac{dx}{dt} = kx(M-x)$



$P(t) = \cancel{\dfrac{MP_0}{P_0+(M-P_0)e^{kMt}}}$

if $x<0$,

$$\frac{dx}{dt} = k\underbrace{x}_{<0}\underbrace{(M-x)}_{>0} < 0$$

if $0<x<M$,

$$\frac{dx}{dt} = k\underbrace{x}_{>0}\underbrace{(M-x)}_{>0} > 0.$$

if $x>M$:

$$\frac{dx}{dt} = k\underbrace{x}_{>0}\underbrace{(M-x)}_{<0} < 0$$

Definition: An equilibrium solution $x = c$ to $\dfrac{dx}{dt} = f(x)$

is **stable** if:

Solutions x(t) starting near c end up staying near c in the long run.

Definition: An equilibrium solution $x = c$ to $\dfrac{dx}{dt} = f(x)$

is **unstable** if it is not stable.

Using the Phase diagram, you can quickly see what effect the initial conditions have on long term behaviour.

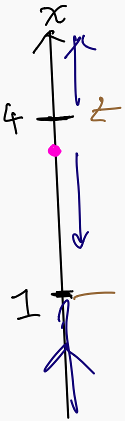
Example: What are the equilibrium solutions? Are they stable?

$$\frac{dx}{dt} = x^2 - 5x + 4$$

Equilibrium solutions $\left(\frac{dx}{dt} = 0\right)$:

$$x^2 - 5x + 4 = 0 \Rightarrow (x-1)(x-4) \Rightarrow \boxed{x=1, x=4}$$

Phase Diagram:



← semi stable.

stable.

if $x < 1$,

$$\frac{dx}{dt} = \underbrace{(x-1)}_{<0}\underbrace{(x-4)}_{<0} > 0.$$

if $1 < x < 4$

$$\frac{dx}{dt} = \underbrace{(x-1)}_{>0}\underbrace{(x-4)}_{<0} < 0$$

if $x > 4$

$$\frac{dx}{dt} = \underbrace{(x-1)}_{>0}\underbrace{(x-4)}_{>0} > 0.$$

Example: Let k, M, h be constant.

What are the equilibrium solutions?

Are they stable?

harvesting at constant.

$$\frac{dx}{dt} = kx(M-x) - h = -k(x-N)(x-H)$$

Do the fish go extinct?

Equilibrium Solution.

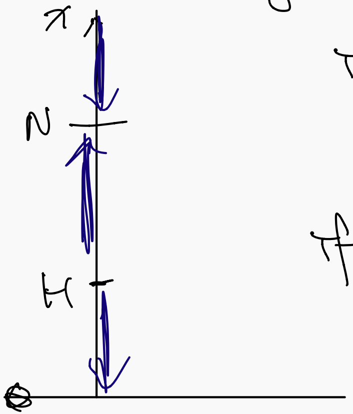
$$\frac{dx}{dt} = 0 \Rightarrow kxM - kx^2 - h = 0$$

$$kx^2 - kMx + h = 0$$

So $$x = \frac{kM \pm \sqrt{k^2M^2 - 4hk}}{2k}$$

Call them N, H, where $N > H$.

Phase diagram.



If $x < H$:

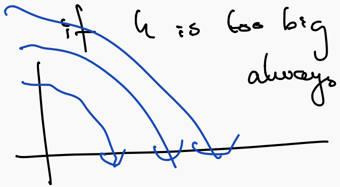
$$\frac{dx}{dt} = -k\underbrace{(x-N)}_{<0}\underbrace{(x-H)}_{<0} < 0.$$

If $H < x < N$

$$\frac{dx}{dt} = -k\underbrace{(x-N)}_{<0}\underbrace{(x-H)}_{>0} > 0$$

If

$$\frac{dx}{dt} = -k\underbrace{(x-N)}_{>0}\underbrace{(x-H)}_{>0} < 0.$$



if h is too big, always become extinct.

Consider: $k=1$ $M=4.$

$$\frac{dx}{dt} = x(4-x) - h$$

$0 = 4x - x^2 - h \Rightarrow x^2 - 4x + h = 0$

How does the qualitative nature of the solutions change when we modify h?

$x = \dfrac{4 \pm \sqrt{16-4h}}{2}$

https://www.desmos.com/calculator/u3x5w62sde

One interpretation: How does the number of equilibrium points depend on h?

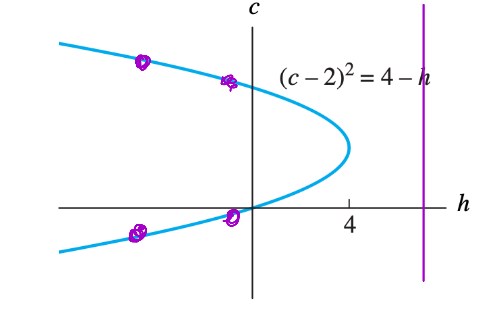


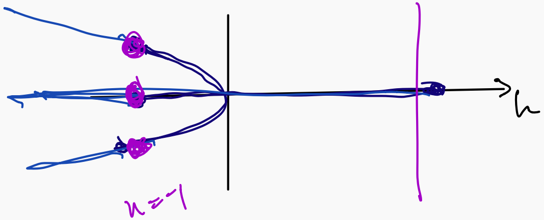
FIGURE 2.2.12. The parabola $(c-2)^2 = 4 - h$ is the bifurcation diagram for the differential equation $x' = x(4-x) - h$.

A bifurcation point for a parameter is a point at which the number of equilibrium solution changes.

Bifurcation diagram is a plot of parameter vs. the location of the equilibrium points.

Here it was at $h=4$.

(equilibria are at $x = \dfrac{4 \pm \sqrt{16-4h}}{2}$)



How many equilibria at $h=-1$?

a) 0
b) 1
c) 2
d) 3