MAT303: Calc IV with applications

Lecture 6 - February 22 2021

Last time:

- Ch 1.6 Substitution methods:
  - $\frac{dy}{dx} = F(ax + by + c)$ $\leftarrow$ $v = ax + by + c$
  - $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ $\leftarrow$ $v = y/x$
  - $\frac{dy}{dx} + P(x)y = Q(x)y^n$ $\leftarrow$ $v =$

Exact Equations.

Today:

- A few more substitutions (Reducible)
- Ch 2.1 (general) population models
  - $\frac{dP}{dt} = (\beta - \delta)P$
- Ch 2.2 Equilibrium solutions

By changing the functions β and δ we can model a large variety of phenomena.

no 'x'.

Example: Solve $yy'' = (y')^2$ for y.

Substitution:

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

This substitution works whenever you want to solve an equation of the form $F(y, y', y'') = 0$.

no 'x'.

Example: Solve $yy'' = (y')^2$ for y.

Substitution:

$$p = y' = \frac{dy}{dx},$$

$yy'' = p^2$

$y'' = \frac{dy'}{dx} = \frac{dp}{dx}$

$= \frac{dp}{dy}\frac{dy}{dx}$

$= \frac{dp}{dy} y' = \frac{dp}{dy} p.$

$y p \frac{dp}{dy} = p^2$

$\frac{1}{p} dp = \frac{1}{y} dy$

$\Rightarrow \log|p| = \log|y| + C$

$\Rightarrow |p| = e^C |y|$

$\Rightarrow p = \pm C y.$

$y' = Cy.$

So (by separation of variables).

$y = De^{Cy}$ For any D, C.

Verify:

$y'' = C^2 D e^{Cy}$ $y' = DCe^{Cy}.$

So

$yy'' = C^2 D^2 e^{2Cy}$ $(y')^2 = D^2 C^2 e^{2Cy}.$

Works for any C, D>0.

This substitution works whenever you want to solve an equation of the form $F(y, y', y'') = 0$.

There is another useful substitution when $F(x, y', y'') = 0$., see textbook Ch1.6.

2.1: Population models

b
d.

General population model:

$\frac{dP}{dt} = (\beta - \delta)P$

$= \beta P - \delta P$ = births per unit of time − deaths per unit of time

* β: birth rate – number of births per unit of time per unit of population.

* δ: death rate – number of death per unit of time per unit of population.

$\frac{dP}{dt} = \frac{1}{2} P$

β

$\frac{dP}{dt} = \frac{1}{100} P$

β.

These can depend on population. (might not be constants).

General population model:

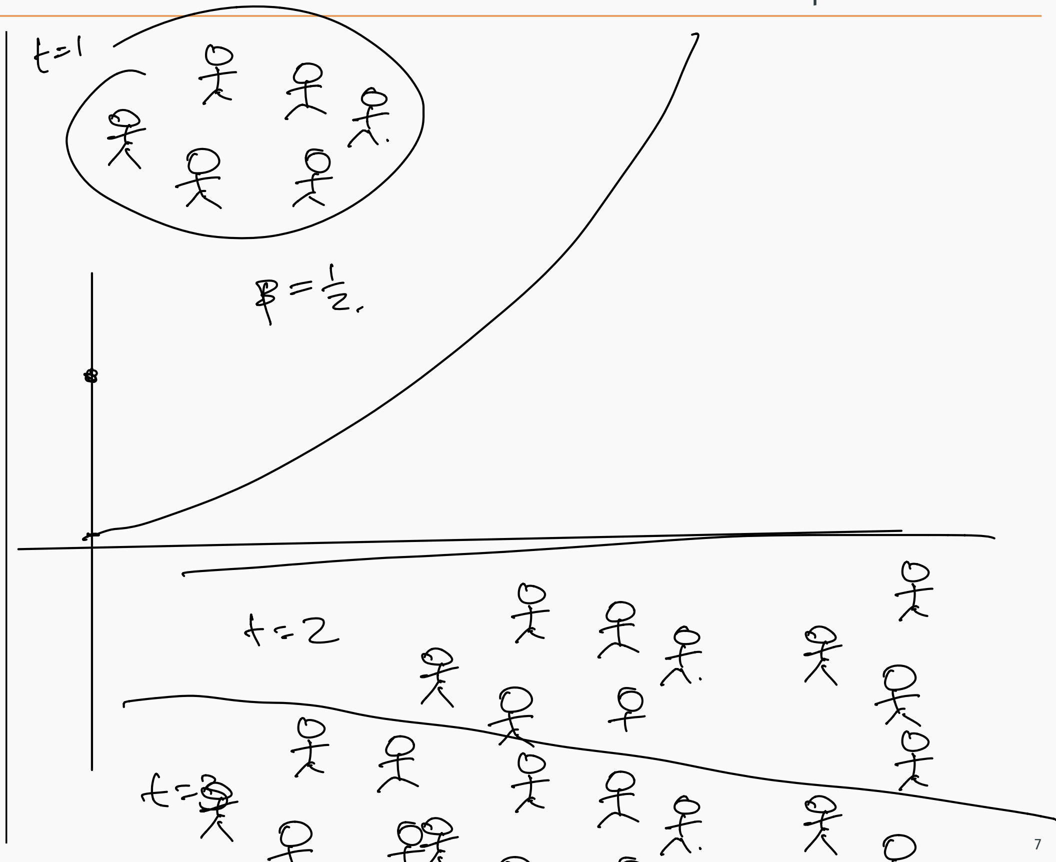
$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume
- Death rate is 0 $\quad \delta = 0.$
- Birth rate is constant $\quad \beta$ const.

Then

$$\frac{dP}{dt} = \beta P.$$

$$P(t) = P(0)e^{\beta t}$$



General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Example 1 Suppose that an alligator population numbers 100 initially, and that its death rate is $\delta = 0$ (so none of the alligators is dying). If the birth rate is $\beta = (0.0005)P$—and thus increases as the population does—then Eq. (1) gives the initial value problem

Assume
- Death rate is 0 $\leftarrow \delta = 0.$
- Birth rate is proportional to population

$$\beta = kP.$$

$$\frac{dP}{dt} = kP^2.$$

Separable:

$$\frac{1}{P^2}\,dP = k\,dt$$

$$-P^{-1} = kt + C$$

$$P = \frac{1}{-kt - C}$$

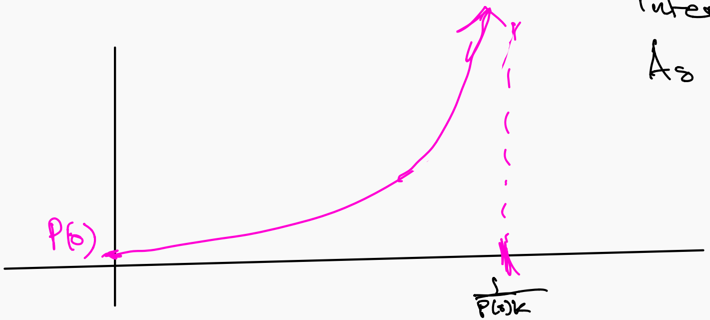
$$P = \frac{1}{C - kt}.$$

if $P(0)$ is initial pop,

$$P(0) = \frac{1}{C} \quad \text{so} \quad C = \frac{1}{P(0)}$$

So

$$P(t) = \frac{1}{\frac{1}{P(0)} - kt} = \frac{P(0)}{1 - P(0)kt}.$$



Interesting: As $t \to \frac{1}{P(0)k}$

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

6×2

Assume

- Constant death rate ← δ const.
- Birth rate is linear and decreasing with respect to population.

$$\beta = \beta_0 - \beta_1 P \quad , \beta_0, \beta_1 > 0.$$

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta)\, P$$

$$= \underbrace{(\beta_0 - \delta)}_{a} P - \underbrace{\beta_1}_{b} P^2.$$

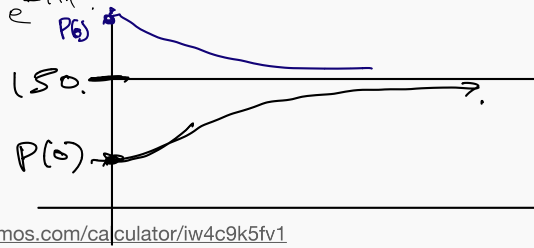
$$\frac{1}{aP - bP^2}\, dP = dt$$

$$\frac{1}{aP\left(1 - \frac{b}{a}P\right)}\, dP = dt$$

$$= \left(\frac{1}{P} + \frac{b}{A - BP}\right) dP = dt$$

$$\log P + \frac{1}{-b} \log(a - b P) = t + c.$$

$$P(t) = \frac{150\, P(0)}{P(0) + \underbrace{(150 - P(0))}_{\text{negative.}} e^{-At}}.$$



Interactive Graph: https://www.desmos.com/calculator/iw4c9k5fv1

Notice:

- No matter what initial condition, the limiting population is

$$\lim_{t\to\infty} P(t) = 150.$$

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume

- Constant death rate
- Birth rate is linear and decreasing with respect to population.

$$\frac{dP}{dt} = kP(M - P)$$

Solution (next HW):

$$P(t) = \frac{MP_0}{P_0 + \left(M - P_0\right)e^{kMt}}$$

Notice:

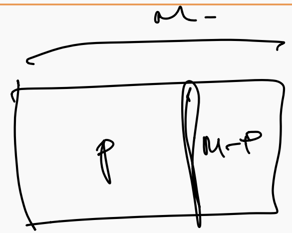
- No matter what initial condition, the limiting population exists
- Limit population is $\lim_{t\to\infty} P(t) = M$.

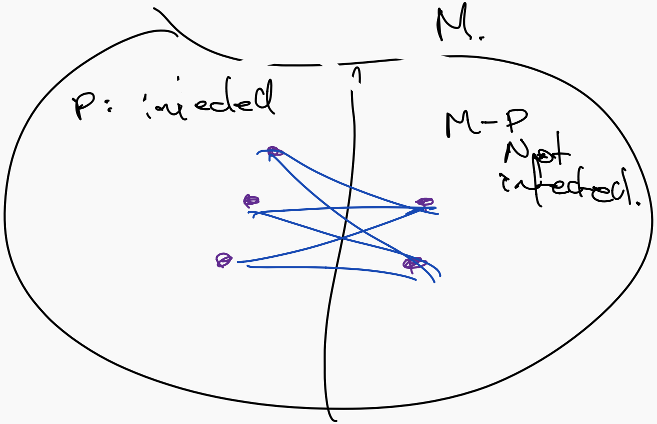
free capacity.

1. Limited environment:
   - Suppose environment can only support up to M individuals.
   - Growth rate $\beta - \delta$ is proportional to $M - P$. $= k(M-P)$
   - Then $\dfrac{dP}{dt} = (\beta - \delta)P = kP(M - P)$.
2. Competition situation:
   - Suppose birth rate constant, but deaths result from chance encounters between the members of the population.
   - Then $\dfrac{dP}{dt} = (\beta - \alpha P)P = kP(M - P)$. $= kPM - kP^2$.
3. Suppose we are modeling disease spread in a constant size population M.
   - This time, $P(t)$ represents the number of infected individuals.
   - Reasonable to assume that rate of new infections is proportional to $P(M - P)$.
   - Thus $\dfrac{dP}{dt} = kP(M - P)$ again.

The same equation can have multiple interpretations.

Different models can give rise to the same equation.





General population model: (Ex 7)

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume

- Death rate is constant ← δ const.
- Birth rate is proportional to ~~time~~ population.

$$\beta = kP$$

$$\frac{dP}{dt} = (kP - \delta)P = kP^2 - \delta P$$

$$= kP\left(P - \frac{\delta}{k}\right)$$

$$\frac{dP}{dt} = kP\left(P - \frac{\delta}{k}\right)$$

$$\frac{1}{P\left(P - \frac{\delta}{k}\right)}\, dP = k.$$

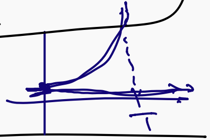
$$-\frac{k}{\delta}\left(\frac{1}{P} - \frac{1}{P - \frac{\delta}{k}}\right) dP = k\, dt$$

$$-\frac{\cancel{k}}{\delta}\left(\log|P| - \log\left|P - \frac{\delta}{k}\right|\right) = \cancel{k}t + C.$$

$$\left(\log|P| - \log\left|P - \frac{\delta}{k}\right|\right) = -\delta t + C.$$

$$\Rightarrow \frac{P}{P - \frac{\delta}{k}} = Ce^{-\delta t} \qquad \frac{P(0)}{P(0) - \frac{\delta}{k}} > 1.$$

Plugging into $t = 0 \Rightarrow \boxed{\frac{P(0)}{P(0) - \frac{\delta}{k}} = C.}$



$$\Rightarrow P = Ce^{-\delta t}P - \frac{\delta}{k}Ce^{-\delta t}$$

$$P(1 - Ce^{-\delta t}) = -\frac{\delta}{k}Ce^{-\delta t}$$

$$\Rightarrow P(t) = \frac{-\frac{\delta}{k}Ce^{-\delta t}}{1 - Ce^{-\delta t}}$$

if $C > 1$, at some time T, $Ce^{-\delta t} = 1$. At that time, $P(T) = \infty$.

if $C < 1$, denominator → 1 as $t \to \infty$, but numerator goes to 0. So.

Sometimes useful for modeling certain endangered species.