MAT303: Calc IV with applications

Lecture 5 - February 17 2021

Today:

- Ch 1.6 Exact differential equations and

  Substitution methods

- Changing variables is a common operation
  - Can make the equation solvable/simpler
  - Can give more insight into the DE
  - Can demonstrate similarities between DEs.

From before,
We know how to solve:

* $\frac{dy}{dx} = f(x)$
* $\frac{dy}{dx} = f(x)\,g(y)$ Separable
* $\frac{dy}{dx} + p(x)y = q(x)$ Linear

a) Yes
b) No.

Consider a function $f(x, y)$ of two variables.

$\frac{\partial}{\partial x}$ means "differentiate with respect to y"

$\frac{\partial}{\partial y}$ means "differentiate with respect to y"

Example: $\cancel{f(x)}$ $f(x,y)$ $= x^2 y + \sin(x)$

Then $\frac{\partial}{\partial y} f = x^2 + 0$

And $\frac{\partial}{\partial x} f = 2xy + \cos x$.

More notation:

f_x means $\frac{\partial}{\partial x} f$ $\qquad f_{xy}$ means $\frac{\partial}{\partial y}\frac{\partial}{\partial x} f$

f_y means $\frac{\partial}{\partial y} f$ $\qquad f_{xx}$ means $\frac{\partial}{\partial x}\frac{\partial}{\partial x} f$

Higher order derivatives:

$$\frac{\partial}{\partial x}\underbrace{\frac{\partial}{\partial y} f}_{x^2} = 2x.$$

$$\frac{\partial}{\partial x}\overbrace{\frac{\partial}{\partial x} f} = 2y - \sin(x)$$

$$\frac{\partial}{\partial y}\underbrace{\frac{\partial}{\partial x} f}_{2xy + \cos x} = 2x$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f = 0.$$

Notice:

$$f_{xy} = f_{yx}$$

This is true in general. It's called Clairaut's theorem.

One variable:

$y = f(g(t))$

If $y = f(g)$

$$\frac{dy}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

$=$

Example:

$y = g(t)^2$

$\frac{dy}{dt} = 2g(t) \cdot g'(t)$.

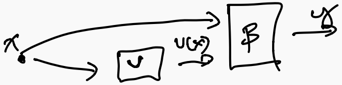
Change in $f(g(t))$

= change in $g(t)$

× rate of change of f w.r.t. g.

If $y = \beta(x, v)$, $v = v(x)$

$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial v}\frac{\partial v}{\partial x}$$



If $y = g(w, z)$

$$\frac{dy}{dx} = g_w w_x + g_z z_x$$

If $y = g(x, y, z)$

$$\frac{dy}{dt} = g_x x_t + g_y y_t + g_z z_t$$

Example:

$\frac{dv}{dx} \cdot \frac{d\beta}{dv}$

$$\frac{d}{dx} x \sin(x)^2 = .$$

Here, $\beta(x, v) = x v^2$, $v(x) = \sin(x)$

By chain rule,

$\frac{dy}{dx} = v^2 + \cos x \cdot 2xv = \sin^2 x + \cos x \cdot 2x \sin x$.

By product rule: $\frac{d}{dx} x(\sin x)^2 = (\sin x)^2 + x\, 2(\sin x)(\cos x)$

Example:

$$f(x, y) = x^y, \quad y = y(x)\,.$$

What is f_x?

Don't try to memorize these exact formulas. Just remember the meaning and reasoning.

Example: Solve

$$\frac{dy}{dx} = (x+y+3)^2$$

Using the substitution $v = x + y + 3$.

$$\frac{dy}{dx} = v^2$$

Need to get rid of y on LHS.

We have

$$y = v - x - 3$$

So $\frac{dy}{dx} = \frac{dv}{dx} - 1$

So

$$\frac{dv}{dx} - 1 = v^2$$

So $\frac{dv}{dx} = v^2 + 1$

$\frac{dy}{dx} = f(x)g(y)$ Separable

$$\frac{1}{v^2+1}\,dv = dx$$

$$\arctan(v) = x + C$$

$$v = \tan(x+C).$$

Find y: $y = \tan(x+C) - x - 3$

Verify:

$$\frac{dy}{dx} = \frac{1}{\cos^2(x+C)} - 1 = \frac{1-\cos^2(x+C)}{\cos^2(x+C)} = \frac{\sin^2(x+C)}{\cos^2(x+C)} = \tan^2(x+C).$$

This works whenever you have a differential equation of the form

$$\frac{dy}{dx} = F(ax + by + c).$$

Let $v = ax + by + c$

$y = \frac{v - c - ax}{b}$

$\frac{dy}{dx} = \frac{1}{b}\frac{dv}{dx} - \frac{a}{b}$

$\frac{dy}{dx} = F(v)$

$$\frac{1}{b}\frac{dv}{dx} - \frac{a}{b} = F(v) \Rightarrow \frac{dv}{dx} = bF(v) + a$$

$v' = bF(v) + a$

Transforming a differential equation like this is very common in applications.

For homogeneous equations like $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$,

the substitution

$$v = \frac{y}{x}$$

is useful.

Example: Solve for y:

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

$$\frac{dy}{dx} = \frac{2x}{y} + \frac{3y}{2x} = F\left(\frac{y}{x}\right)$$

where

$$F(p) = \frac{2}{p} + \frac{3}{2}p$$

Let $v = \frac{y}{x}$

Then

$$\frac{dy}{dx} = \frac{2}{v} + \frac{3}{2}v$$

$$y = vx$$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v \qquad \text{(not } v\text{).}$$

$$\frac{dv}{dx}x + v = \frac{2}{v} + \frac{3}{2}v$$

$$\frac{dv}{dx} = \frac{1}{x}\left(\frac{2}{v} + \frac{1}{2}v\right)$$

$$\frac{dv}{dx} = \frac{1}{x}\left(\frac{2}{v} + \frac{1}{2}v\right)$$

$$\frac{1}{\left(\frac{2}{v} + \frac{1}{2}v\right)} dv = \frac{1}{x} dx$$

$$\frac{2v}{4+v^2} dv = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{2v}{4+v^2} dv = \log|x| + C$$

$$\log|4+v^2| = \log|x| + C$$

$$4 + v^2 = C|x|$$

$$4 + \frac{y^2}{x^2} = C|x|$$

$$\frac{y^2}{x^2} = C|x| - 4$$

$$y^2 = C|x|x^2 - 4x^2.$$

For the Bernoulli equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \Bigg\} \text{ non-linear}$$

You should take $v = y^{1-n}$ which turns it into

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)\,. \Big\} \leftarrow \text{ linear.}$$

Example:

$$2xyy' = 4x^2 + 3y^2$$

Example: $\frac{d}{dx} y^2x = y^2 + 2yy'x$

$$2x + y^2 + 2xyy' = 0$$

Notice that

$$2x+y^2+2xyy' = \frac{d}{dx}(x^2+y^2x)$$

The differential equation becomes

$$\frac{d}{dx}(x^2+y^2x) = 0$$

Solu.

$$x^2+y^2x = C \quad (C \text{ is arbitrary}).$$

Want:

$$y+3xy' =$$

Definition: A DE $M + N\frac{dy}{dx} = 0$ is **exact** if we can find $f(x, y)$ such that ~~$f_x = M$ and $f_y = N$.~~

$$M+N\frac{dy}{dx} = \frac{d}{dx}f$$

Is the following DE exact or not?

$$y+3x\frac{dy}{dx} = \frac{d}{dx}(f(x,y))$$

$$f_x+f_y\frac{dy}{dx} = \frac{d}{dx}(f(x,y))$$

multivariable chain rule

$\boxed{\text{Clairauts: } f_{xy}=f_{yx}}$

$f_{xy}=1 \quad f_{yx}=3$

Is the following DE exact or not?

Test for exactness: A DE $M + N\frac{dy}{dx} = 0$ can only be exact if $M_y = N_x$.

Example:

$$2x + y^2 + 2xyy' = 0$$

Definition: A DE $M + N\dfrac{dy}{dx} = 0$ is **exact** if we can find $f(x, y)$ such that $f_x = M$ and $f_y = N$.

Is the following DE exact or not?

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Test for exactness: A DE $M + N\dfrac{dy}{dx} = 0$ can only be exact if $M_y = N_x$.

If we know an equation is exact, it is easy to figure out what f is.

$$\overbrace{(6xy - y^3)}^{M}\,dx + \overbrace{(4y + 3x^2 - 3xy^2)}^{N}\,dy = 0$$

1) $\overbrace{(6xy-y^3)}^{M} + \overbrace{(4y+3x^2-3xy^2)}^{N}\frac{dy}{dx}=0$

check if it's exact.

a) Yes
b) No.

$M_y = 6x - 3y^2$

$N_x = 6x - 3y^2$

Want:

$\frac{d}{dx}(f(x,y)) = M + N\frac{dy}{dx}$

$\frac{d}{dx}(f(x,y)) = f_x + f_y\frac{dy}{dx}$

a) $3x^2y - xy^3$
b) $3x^2y - xy^3 + 2$
c) $3x^2y - xy^3 + e^4 \sin y$
d) I don't know

1st constraint:

$\frac{\partial f}{\partial x} = f_x = 6xy - y^3$

So

$f(x,y) = 3x^2y - xy^3 + g(y)$ ← arbitrary func. of y.

2nd constraint:

$f_y = \cancel{3x^2 - 3xy^2} + g'(y) = 4y + \cancel{3x^2 - 3xy^2}$

$g'(y) = 4y$

$g(y) = 2y^2 + C$

So $f(x,y) = 3x^2y - xy^3 + 2y^2 + C$.

Definition: A DE $M + N\frac{dy}{dx} = 0$ is **exact** if we can find $f(x, y)$ such that $f_x = M$ and $f_y = N$.

Test for exactness:

exact only if $M_y = N_x$

i.e.

$\frac{d}{dx}(3x^2y - xy^3 + 2y^2) = 0.$

So $3x^2y - xy^3 + 2y^2 = C.$

(implicitly solution).

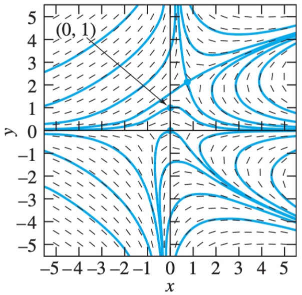

FIGURE 1.6.7. Slope field and solution curves for the exact equation in Example 9.

- Differential equations can often be transformed into different forms
  - Useful to reduce one problem to another problem
  - Useful when using a computer to solve the differential equation

Exact equations:

$$M\,dx + N\,dy = 0$$

1) Check that its exact: $M_y = N_x$

2) Find f such that

$$df = M\,dx + N\,dy \quad (f_x = M,\ f_y = N)$$

3) Solution is

$$f(x,y) = C$$

(implicit solution).

$$M + N\frac{dy}{dx} = 0$$