MAT303: Calc IV with applications

Lecture 4 - February 15 2021

Last time:

- Ch 1.4 Separable equations
    - $\frac{dy}{dx} = f(x)g(y)$ $\Longrightarrow$ $\frac{1}{g(y)}\,dy = f(x)\,dx$
- Solving problems using separable equations. (Applications)
    - Radioactive decay
    - Water escaping from a tank

Why it works:
Chain rule.
See textbook.

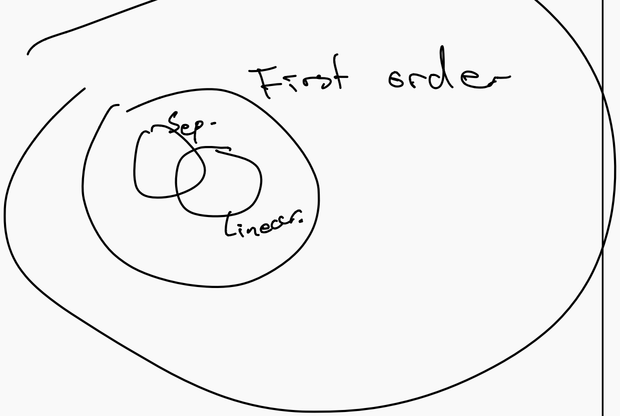
Today:

- Ch 1.5 Integrating Factors for first order linear DEs

Types of DEs (these are not exclusive):

- First order: $\frac{dy}{dt} = f(t, y)$
- First order Separable: $\frac{dy}{dt} = f(t)g(y)$
- First order linear: $\frac{dy}{dx} + p(x)y = q(x)$

DEs.



Classify:

$y' = \underbrace{x}_{f(x)}\underbrace{y}_{g(y)}$

Separable.

Linear: $y' \underbrace{- x}_{p(x)} y = \underbrace{0}_{q(x)}$

$y' = y^2 + x$

Not separable
Not Linear.

$y'' + 2y' + 3y = 0$

Not first order.

$(4 + t^2)\frac{dy}{dt} + 2ty = 4t$

$$\frac{dy}{dt} + \underbrace{\frac{2t}{4+t^2}}_{p(t)} y = \underbrace{\frac{4t}{4+t^2}}_{q(t)}$$

$\frac{dy}{dt} + \underbrace{2}_{p(t)}y = \underbrace{4t}_{q(t)}$

Linear

$\frac{dy}{dt} = 4t - 2y$

Linear.

$$\frac{dy}{dt} + \underbrace{2}_{p(t)}y = \underbrace{4t}_{q(t)}.$$

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Today:

- Ch 1.5 Integrating Factors

Separation of variables does not work:

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

$$\frac{dy}{dt} = \frac{4t - 2ty}{(4+t^2)}$$

Product rule:

$$\frac{d}{dt}(fg) = f'g + fg'$$

E.g:

$$\frac{d}{dt}(t \sin t) = \sin t + t \cos t$$

$$\frac{d}{dt}(t^2 y) = 2ty + t^2 \frac{dy}{dt}$$

If $\dfrac{dy}{dt} = t^2$, what is y?

Ans: $y = t^3 + C$

If $\dfrac{d}{dt}(yt) = t^3$, what is y?

$$yt = \frac{t^4}{4} + C$$

$$y = \frac{t^3}{4} + \frac{C}{t}$$

If $\dfrac{d}{dt}((4+t^2)y) = t$, what is y?

$$(4+t^2)y = \frac{t^2}{2}$$

$$y = \frac{t^2}{2(4+t^2)}.$$

Example: Solve $(4+t^2)\frac{dy}{dt} + 2ty = 4t$

By product rule, (replace LHS).

$$\frac{d}{dt}\left((4+t^2)y\right) = 4t$$

$$\Downarrow$$

$$(4+t^2)y = 2t^2 + C$$

$$\Downarrow$$

$$\boxed{y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}}$$

Key idea: writing LHS as $\frac{d}{dt}(\ldots)$

From previous slides:

Product rule $\quad \frac{d}{dt}((4+t^2)y) = (4+t^2)\frac{dy}{dt} + 2ty$

If $\frac{d}{dt}((4+t^2)y) = 4t$, then $y = \frac{2t^2}{(4+t^2)}$

Example 1: Find the solution to the DE

$$\frac{dy}{dt} \underbrace{- 2}_{p(t)} y = e^{5t}$$

Want to use previous trick: LHS = $\frac{d}{dt}(\ldots)$?

$$\frac{d}{dt}\Big(\qquad \Big) = e^{5t}$$

Impossible

integrating factor

⓪ Multiply both sides by $\mu = e^{-2t}$:

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}y = e^{3t}$$

Then

$$\frac{d}{dt}\left(e^{-2t}y\right) = e^{3t}$$

① Integrate. $e^{-2t}y = \frac{1}{3}e^{3t} + C$

② Solve for y:

$$y = \frac{1}{3}e^{5t} + Ce^{2t}.$$

is the general solution.

③ Verify:

$$\frac{dy}{dt} - 2y = \frac{5}{3}e^{5t} + 2Ce^{2t} - \frac{2}{3}e^{5t} - 2Ce^{2t}$$
$$= e^{5t}$$

How did we know to multiply by $\mu = e^{-2t}$?

Answer: $\mu = e^{\int p dt}$ always works.

why it works:

$$y' + py = q$$

Multiply by $e^{\int p dt}$: $e^{\int p dt}y' + e^{\int p dt}py = q$

$$\left(ye^{\int p dt}\right)' = q$$

$y' + py = q$

How did we know to multiply by $\mu = e^{-2t}$?

Answer: $\mu = e^{\int p dt}$ always works.

Solving linear DEs this way is called the method of integrating factors.

Example: Solve $t\dfrac{dy}{dt} + 2y = 4t^2$

(-1) Get into standard form.

$$y' + \underbrace{\frac{2}{t}}_{p(t)} y = 4t$$

(0) Multiply $\mu = e^{\int p(t)\,dt} = e^{\int \frac{2}{t}\,dt} = e^{2\log|t|} = t^2$

$$t^2 y' + 2ty = 4t^3$$

(1) Write LHS as ( )' :

$$(t^2 y)' = 4t^3$$

(2) Integrate

$$t^2 y = t^4 + C$$

(3) Solve for y: $y = t^2 + \frac{C}{t^2}$.

Standard form:

$$y' + py = q$$

How to come up with:

$\mu = e^{\int p\,dt}$.

$y' + py = q$

Wish:

$\mu y' + p\mu y = (\mu y)'$

$\Rightarrow \mu y' + p\mu y = \mu' y + \mu y'$

$\Rightarrow \mu p y = \mu' y$

$\Rightarrow \mu p = \mu'$.

$\Rightarrow \mu p = \frac{d\mu}{dt}$

$\Rightarrow p\,dt = \frac{1}{\mu}\,d\mu$

$\Rightarrow \int p\,dt \doteq \log|\mu|$

$\mu = e^{\int p\,dt}$.

Solving linear differential DEs with integrating factors:

1. Write DE in ‘standard form’

$$y' + py = q.$$

2. Multiply by integrating factor $\mu = e^{\int p dt}$

$$\mu y' + \mu p y = q\mu$$

3. Rewrite LHS:

$$(y\mu)' = q\mu$$

4. Integrate and solve for y

$$y\mu = \int q\mu$$

$$y = \mu^{-1} \int q\mu$$

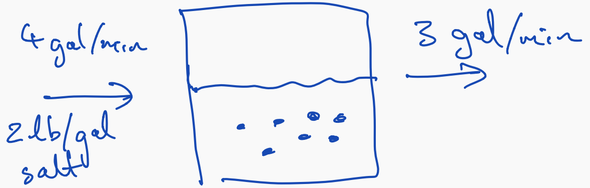
Deriving the method:

1. Wishful thinking:

2. Solve to find the correct expression for μ

Example 4 Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie—caused by past industrial pollution that has now been ordered to cease—is five times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

Example 5 A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?



Q: How much salt at $t=30$

$y(t)$: amount in salt in tank at time t

Goal: $y(30)$

$$\frac{dy}{dt} = \frac{\text{salt in}}{\text{minute}} - \frac{\text{salt out}}{\text{minute}}.$$

$$= \text{concentration in} \times \text{water in} - \text{concentration out} \times \text{water out}$$

water out $\to 3$

$$= 2 \cdot 4 - \frac{y}{90+t} \cdot 3$$

$$\frac{dy}{dt} = 8 - \frac{3y}{90+t}$$

It is linear DE:

$$\frac{dy}{dt} + \frac{3}{90+t} y = 8$$

Multiply by $\mu = e^{\int \frac{3}{90+t} dt} = e^{3\log|t+90|}$
$= (t+90)^3$

$$(t+90)^3 y' + 3(t+90)^2 = 8(t+90)^3$$

$$\left((t+90)^3 y\right)' = 8(t+90)^3$$

$$(t+90)^3 y = 2(t+90)^4 + C$$

$$y = 2(t+90) + \frac{C}{(t+90)^3}.$$

initial condition.

$y(0) = 90$, so $2 \cdot 90 + \frac{C}{90^3} = 90.$

$$\frac{C}{90^3} = -90$$

$$C = -90^4$$

so

$$y(30) = 2(120) + \frac{-90^4}{120^3}.$$

$$= 202\,\text{lb}.$$

Makes sense.

$\pm$

$+$

y

$$y' = f(x,y)$$

Lecture recording will be up soon.

Answering questions until 7:25.