MAT303: Calc IV with applications

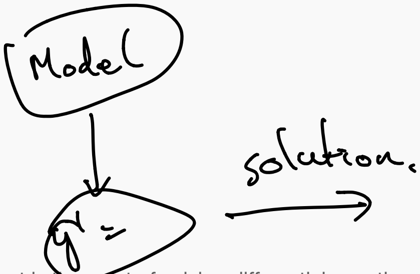
Lecture 3 - February 10 2021

Last time:

- Different ways of interpreting functions and DEs
- Slope field

Today:

- Separable equations
- Solving problems using separable equations. (Applications)



What is the point of solving differential equations:

- Answer questions like
  - What is y(4)?
  - What is y(t)?
- Get an idea of what the behavior of the model
- See what happens when initial condition changes
- See what happens in the long run
- See what happens when model changes

First order equation where RHS does not depend on y:

$$\frac{dy}{dx} = -6x$$

$$y(x) = -3x^2 + C$$

Verify:

$$y' = -6x$$

First order equation where RHS does depend on y:

Solve for y: $\frac{dy}{dx} = -6xy$

New technique: separation of variables

$$\frac{1}{y}\frac{dy}{dx} = -6x$$

$$\frac{1}{y}\,dy = -6x\,dx$$

$$\int \frac{1}{y}\,dy = \int -6x\,dx$$

$$\log|y| + C = -3x^2 + D$$

algebraic manipulation

$$\log|y| = -3x^2 + E$$

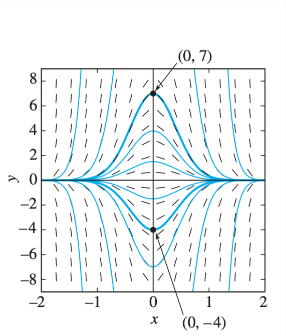
$$|y| = e^{-3x^2+E}$$

$$y = \pm e^{-3x^2+E}$$

$$y = \pm e^{E} e^{-3x^2}$$

$$y = \pm G\, e^{-3x^2}$$

$$y = F e^{-3x^2}, \quad F \in \mathbb{R}$$



Verify:

$$\frac{dy}{dx} = F(-6x)e^{-3x^2}$$

$$= (-6x)\,Fe^{-3x^2}$$

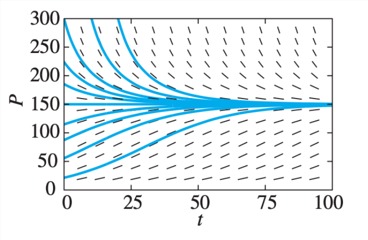
$$= -6xy.$$

Example 2: logistic population growth
(From lecture 2), HW)

$$\frac{dP}{dt} = kP - lP^2, \quad dt$$

$k = 0.0225, \quad l = 0.0003$

$P(0) = 25$



$$\frac{1}{kP - lP^2}\frac{dP}{dt} = 1$$

$$\frac{1}{kP - lP^2}\, dP = dt$$

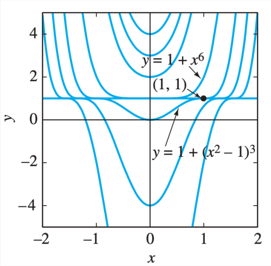
$$\underbrace{\int \frac{1}{kP - LP^2}\, dP}_{\text{Partial fractions.}} + D = t + C$$

Example 3: from textbook

Solve for y:

$$y' = 6x(y-1)^{2/3}$$

$y(0) = 1$



$$\frac{dy}{dx} = 6x(y-1)^{2/3}$$

$$\int (y-1)^{-\frac{2}{3}}\, dy = \int 6x \, dx$$

$$3(y-1)^{1/3} = 3x^2 + C$$

$$(y-1)^{1/3} = x^2 + C$$

$$y - 1 = (x^2+c)^3$$

$$\boxed{y = (x^2+c)^3 + 1 \quad c \in \mathbb{R}}$$

$$1 = (0^2 + c)^3 + 1$$

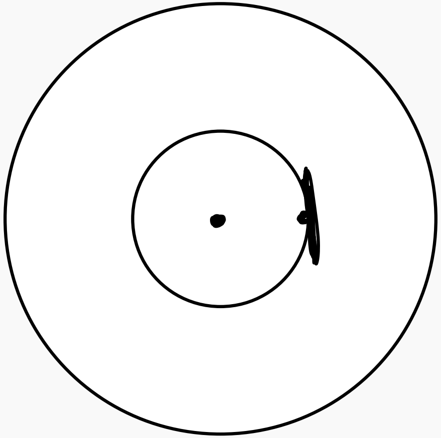
$$0 = c^3$$

$$c = 0.$$

$y =$

$x^2 + y^2 = C$

$f(x,y) = x^2 + y^2$



$2x + 2yy' = 0$

$\boxed{yy' = -x}$

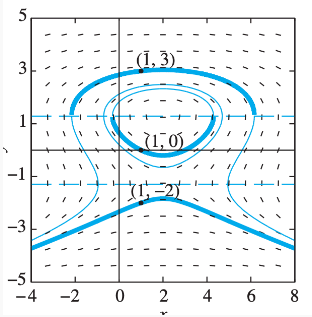
$x = 1, \quad y = 0.$

$0y' = 1$

$y' = \frac{1}{0}.$

Example 3 (from textbook)

$$y' = \frac{4-2x}{3y^2-5}$$



$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx$$

$$y^3 - 5y = 4x - x^2 + C$$

For each $C \in \mathbb{R}$,

$$y^3 - 5y - 4x + x^2 = C$$

defines an implicit soln.

$y =$ __________

At $x=1$,

$$y^3 - 5y - 4 + 1 = C$$

$$y^3 - 5y = C + 3$$

$y =$ __________

When does separation of variables work?

a) $\frac{dy}{dx} = ye^x$ ✓ $\quad \frac{1}{y}\,dy = e^x\,dx$

b) $\frac{dy}{dx} = \cos(y)\sin(x)$ ✓ $\quad \frac{1}{\cos y}\,dy = \sin x\,dx$

c) $y' = \cos(e^x + y)$ ✗

d) $y' = y$ ✓

e) $\cancel{y' = \frac{4-2x}{3y^2-5}}$

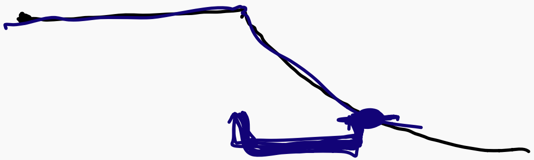
Example applications:

Radioactive decay:
Let N(t) be the number of atoms of a certain radioactive isotype at time t.
Then:

$$\frac{dN}{dt} = -kN$$

This is used for radiocarbon dating:

- radioactive carbon stays constant for living organism
- When organism dies, this decays according to the differential equation above
- So we if we measure the radioactive carbon content of an object, we can figure out how long ago it died.



Example:

- Charcoal at Stonehenge contains 63% as much radioactive carbon compared to present-day charcoal of the same mass.
- How old is the sample? ← T
- Assume that radioactive carbon decays according to the differential equation

$$\frac{dN}{dt} = -kN, \quad k = 0.0001216$$

Solution:

* Solve for N(t), amount of carbon at t
* $N(T) = 0.63\,N(0)$
* Let t=0 be the time that the charcoal died.

$$\frac{1}{N}\,dN = -k\,dt$$

$$\log|N| = -kt + C$$

$$|N| = e^{-kt+C}$$

$$= e^{C} e^{-kt}$$

$$N = \pm e^{C} e^{-kt}$$

$$N(t) = A e^{-kt}$$

$$N(T) = N(0)\,0.63$$

$$A e^{-kT} = A\,0.63$$

$$e^{-kT} = 0.63$$

$$-kT = \log 0.63$$

$$T = \frac{-\log 0.63}{k}$$

$= 3799$ years.

- Suppose you have water in a tank, of depth y
- Tank has a hole at the bottom.
- Then the velocity of the water flow at the bottom is

$$v = \sqrt{2gy}$$

↖ 32ft/s

Question: Suppose you have a hemispherical bowl has radius 4ft

and it is full of water. Suppose you open up a hole of diameter 1in at time $t = 0$.

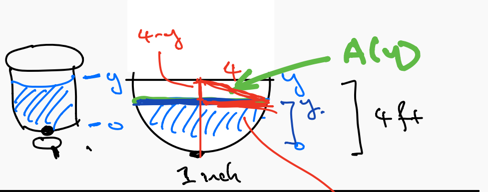
1. Write down a differential equation for the water depth y.
2. What is the depth of water at time t? ← $y(0) = 4$
3. How long does it take the tank to empty?

Key idea: we need to find a differential equation for the volume V.

Rate of change in volume = cross sectional area * velocity

$$\frac{dV}{dt} = -a \cdot \sqrt{2gy}$$

↑ area of hole: $= \pi\left(\frac{1}{24}\right)^2$



y: depth of water.
V: volume of water.

$$\frac{dV}{dt} = A(y)\frac{dy}{dt}$$

$$1.\ \boxed{-a \cdot \sqrt{2gy} = \frac{dy}{dt}\,\pi\left(16-(4-y)^2\right).}$$

$$2\text{-}\ 1\,dt = \frac{\pi\left(16-(4-y)^2\right)}{a(2gy)^{1/2}}\,dy$$

$$t + c = -72\left(\frac{16}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right)$$

* Find c by plugging in $t=0$, $y=4$.

* Find time to empty by plugging $y=0$.

End of Ch. 4

See textbook and hw for other applications such as

- **Drug elimination**
- **Population Growth**
- **Newton's law of cooling**
- **Compound interest**
- **Water flow in a tank**
- **Logistic population growth**
- **Radioactive decay**