MAT303: Calc IV with applications

Lecture 25 - May 5 2021

Recently: Solutions homogeneous constant coefficient systems:

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

And: Solutions to nonhomogeneous systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

Today: geometric interpretation of eigenvectors (Ch 5.3)

We will deal only with $n = 2$ case.

We've been solving systems such as

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned} \tag{27}
$$

Or equivalently

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + f(t)$$

The solution is a collection of functions $x_1(t), \ldots, x_n(t)$, or equivalently a vector function $\mathbf{x}(t)$

$$\frac{dx}{dt} = 3x - 2y \qquad \frac{dy}{dt} = -\frac{1}{2}x \Rightarrow x' = \begin{bmatrix} 3 & -2 \\ -\frac{1}{2} & 0 \end{bmatrix} x$$

E.g.: $\dfrac{dx}{dt} = -2y, \quad \dfrac{dy}{dt} = -\dfrac{1}{2}x.$ $\Rightarrow x' = \begin{bmatrix} 0 & -2 \\ -\frac{1}{2} & 0 \end{bmatrix} x$

Could describe a fight between two populations 'x' and 'y'

Who wins?

Recall from your earlier education that such objects can be viewed as *parametric curves* in $\mathbb{R}^n$.

E.g. $\mathbf{x}(t) = (\cos t, \sin t)$ traces out a circle in $\mathbb{R}^2$.

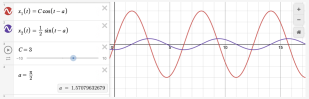
So a solution to a system of DEs can be viewed as a parametric curve. See lecture 17 for more on this.

Method 1: Turning a system into a higher order equation

Consider the system

$$x'(t) = -2y(t)$$
$$y'(t) = \frac{1}{2}x(t)$$

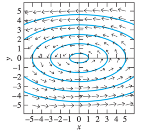
Solution: $x(t) = C\cos(t-\alpha), \quad y(t) = \frac{1}{2}C\sin(t-\alpha)$



There is another way to think about the solution.

The state of the system is described by a path $(x(t), y(t))$ through the plane $\mathbb{R}^2$.

For this DE, the solution curves trace out ellipses in the state space.



<- Phase plane portrait

FIGURE 4.1.6. Direction field and solution curves for the system $x' = -2y$, $y' = \frac{1}{2}x$ of Example 6.

Make sure you understand how this different the direction fields we considered in Ch1. There is no time axis.

Test your understanding:
- Which curve corresponds to the solution on the left?
- What is the effect does changing C have?
- What is the effect does changing α have?
- How did we draw the direction field?
- How can we determine that the trajectories are actually ellipses?

9

We covered how to solve these systems using the eigenvector method.

Today we'll see how the eigenvectors give us information about the solution.

We've been solving systems such as

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
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Or equivalently

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + f(t)$$

The solution is a collection of functions $x_1(t), \ldots, x_n(t)$, or equivalently a vector function $\mathbf{x}(t)$

Recall from your earlier education that such objects can be viewed as *parametric curves* in $\mathbb{R}^n$.

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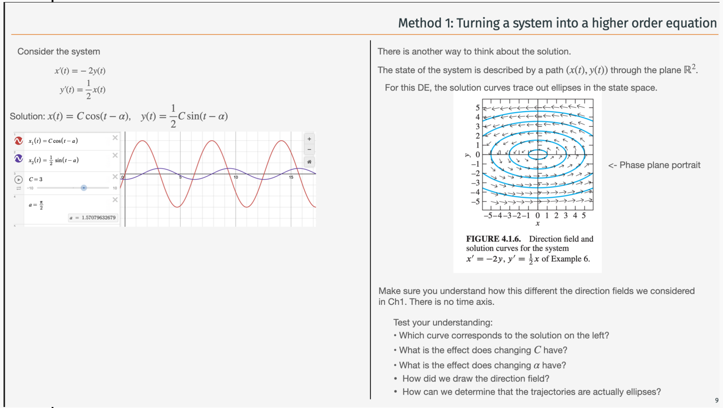
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We covered how to solve these systems using the eigenvector method.

Today we'll see how the eigenvectors give us information about the solution.

Example 1: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -2, \quad \lambda_2 = 5.$

The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

General solution:

$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$x(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 6 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

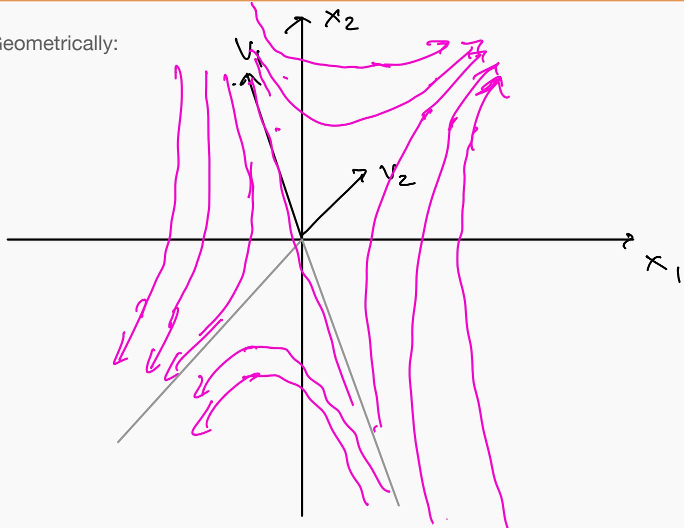
$= \begin{bmatrix} -c_1 e^{-2t} + c_2 e^{5t} \\ 6c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}.$

c_1, c_2 depend on initial conditions

$(-c_1 e^{-2t} + c_2 e^{5t}, \; 6c_1 e^{-2t} + c_2 e^{5t})$

~~Note that $\mathbf{x}(0) = (c_1, c_2)$.~~

Geometrically:



Consider $t \to \infty$.

$x(t) \approx c_2 e^{\lambda_2 t} v_2$

Consider $t \to -\infty$.

$x(t) \approx c_1 e^{\lambda_1 t} v_1$

The origin is called a **saddle point** for the system.

Solns do not cross lines because:

* 2 solns never cross, by uniqueness of solns to DE
* The lines themselves are solns ($c_1=0$ or $c_2=0$).

Example 2: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -14, \quad \lambda_2 = -7$

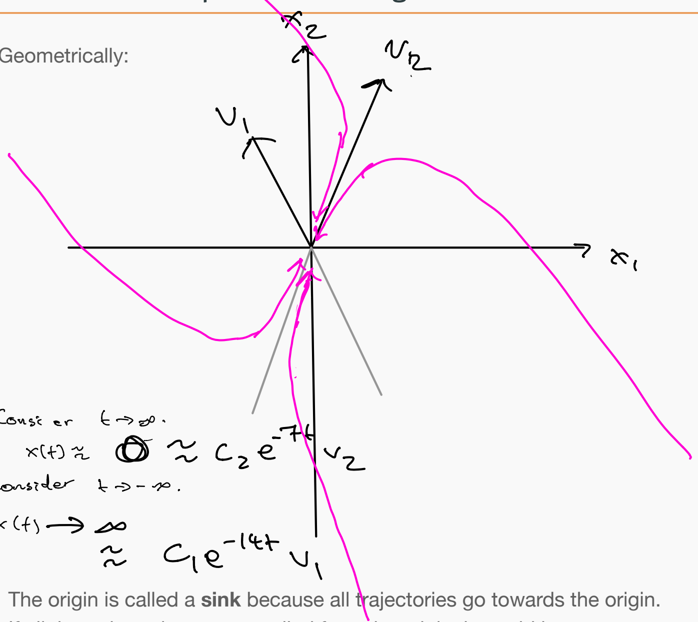
The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

General solution:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$x(t) = c_1 e^{-14t} v_1 + c_2 e^{-7t} v_2.$$

Geometrically:



Consider $t \to \infty$.

$x(t) \approx 0 \approx c_2 e^{-7t} v_2$

Consider $t \to -\infty$.

$x(t) \to \infty$

$\approx c_1 e^{-14t} v_1$

- The origin is called a **sink** because all trajectories go towards the origin.
- If all the trajectories were repelled from the origin, it would be a **source.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal sink.**

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = -\begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -2 & 13 \end{bmatrix}$$

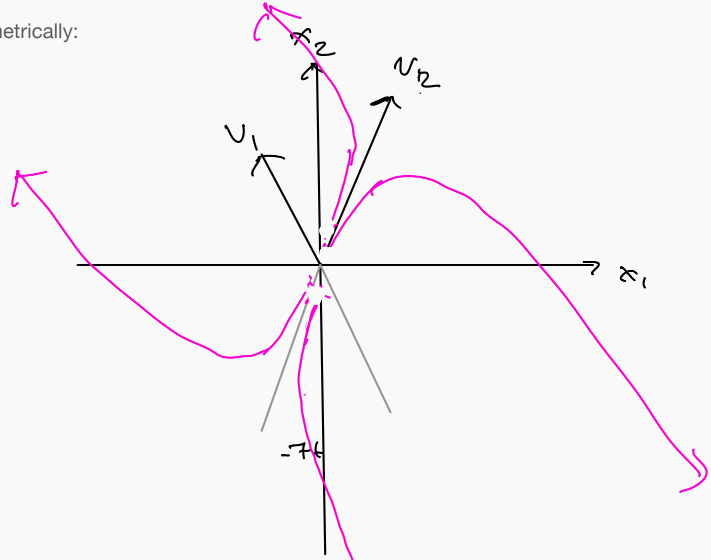
The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 14, \quad \lambda_2 = 7$

The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

This system is just the time reversal of the previous system.
Notice the eigenvalues are the same, but eigenvalues have opposite signs.

General solution:

Geometrically:



- The origin is called a **source** because all trajectories go towards the origin.
- If all the trajectories were attracted to the origin, it would be a **sink.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal source.**

λ_2 0 λ_1 λ_1 λ_2 0

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} -36 & -6 \\ 6 & 1 \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -35$, $\lambda_2 = 0$

The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

General solution:

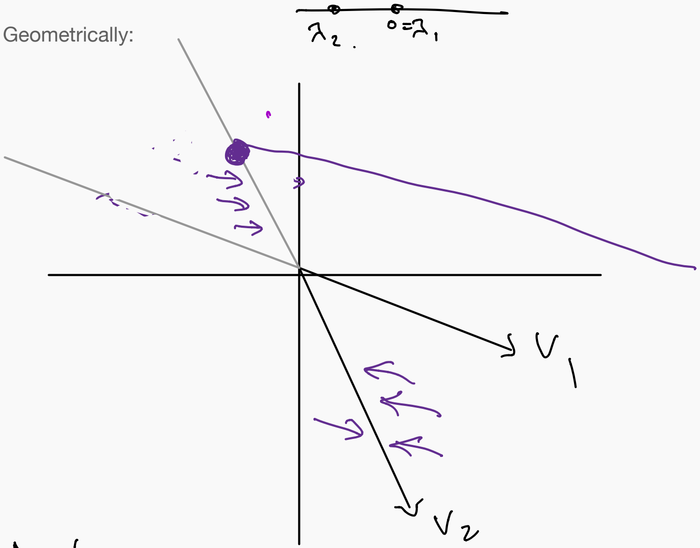
$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 v_2.$$

* So all trajectories are parallel to v_1.

* all trajectories are attracted to the line $c_2 v_2$.

↑ This can be used to determine whether the trajectories follow v_1 or $-v_1$.

Geometrically:



As $t \to \infty$ $x(t) \approx c_2 v_2$. because $\lambda_1 < 0$.

- If we take the time reversal, trajectories reverse direction. Solutions are repelled from the line.

In other words, $x(t)$ is attracted to the line $c_2 v_2$.

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$

The eigenvalue of $\mathbf{A}$ are $\lambda = 2$ (repeated)

A choice of linearly independent eigenvectors is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

General solution:

Geometrically:

- The origin is called a **source** because all trajectories go towards the origin.
- If all the trajectories were attracted to the origin, it would be a **sink.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal source.**

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$

The eigenvalue of $\mathbf{A}$ are $\lambda = 2$ (repeated)

There is only one eigenvector,
however we have also have a generalized eigenvector:

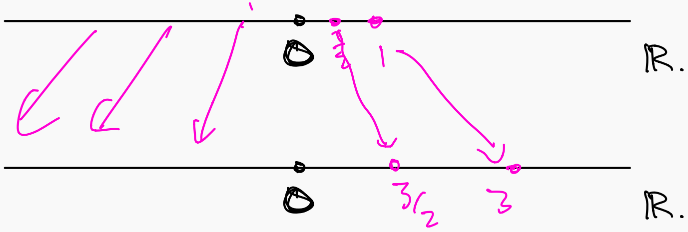
$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

General solution:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}.$$

Geometrically:

In one dimension, multiplying by $\lambda \in \mathbb{R}$
can be thought of as stretching or reflection the real line:



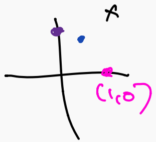
$f(x) = 3x$

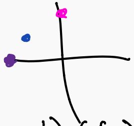
$f(x) = -1$ is reflection.

In two dimensions, multiplying by a matrix $\mathbf{A}$
can be thought of as stretching or rotation or shearing of the plane:

E.g.

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ corresponds to rotation by $\pi/2$

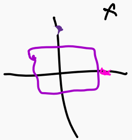


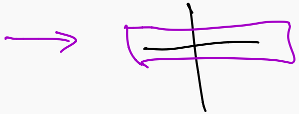


$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$ corresponds to stretching in the x-direction

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$$





So if $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ parameterizes some curve,

then $\mathbf{A}\mathbf{x}(t)$ parameterizes a stretching/rotation/shearing of that curve.

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda = \pm\, 10i$

The eigenvectors are

$$\begin{bmatrix} 3+5i \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-5i \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} i$$

General solution:

$$x(t) = c_1\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\cos 10t - \begin{bmatrix} 5 \\ 0 \end{bmatrix}\sin 10t\right) + c_2\left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}\cos 10t + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\sin 10t\right).$$

$$= \begin{bmatrix} (3c_1+5c_2)\cos 10t + (-5c_1+3c_2)\sin 10t \\ 4c_1 \cos 10t + 4c_2 \sin 10t \end{bmatrix}$$

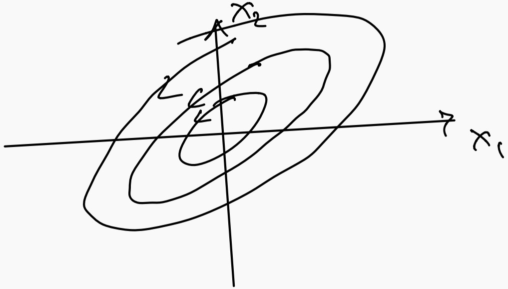
$$= \underbrace{\begin{bmatrix} 3c_1+5c_2 & -5c_1+3c_2 \\ 4c_1 & 4c_2 \end{bmatrix}}_{P} \begin{bmatrix} \cos 10t \\ \sin 10t \end{bmatrix}$$

Need the following fact from linear algebra:
matrix multiplication can be geometrically interpreted as a rotation+stretch+shear.

Geometrically:

Now $\begin{bmatrix} \cos 10t \\ \sin 10t \end{bmatrix}$ parameterizes the unit circle.

So $P\begin{bmatrix} \cos 10t \\ \sin 10t \end{bmatrix}$ is a rotation/stretching/shearing of a circle.



Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -17 \\ 8 & -7 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda = -1 \pm 10i$

The eigenvectors are

$$\begin{bmatrix} 3+5i \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-5i \\ 4 \end{bmatrix}$$

General solution:

$$x(t) = c_1\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\cos 10t - \begin{bmatrix} 5 \\ 0 \end{bmatrix}\sin 10t\right) + c_2\left(\begin{bmatrix} 5 \\ 0 \end{bmatrix}\cos 10t + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\sin 10t\right).$$

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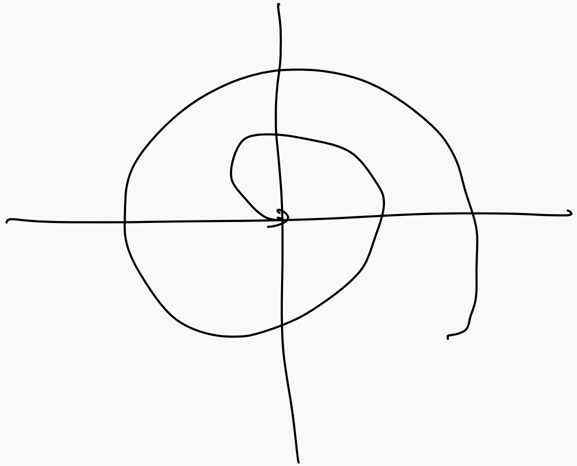

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