MAT303: Calc IV with applications

Lecture 24 - May 3 2021

Recently: Solutions homogeneous constant coefficient systems:

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

Today: Solutions to nonhomogeneous systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

$$x' = P(t)\, x + f(t)$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

$x' = P(t)\,x$

homogeneous.

Principle of superposition:

THEOREM 4 Solutions of Nonhomogeneous Systems

Let $\mathbf{x}_p$ be a particular solution of the nonhomogeneous linear equation in (47) on an open interval I on which the functions $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be linearly independent solutions of the associated homogeneous equation on I. If $\mathbf{x}(t)$ is any solution whatsoever of Eq. (47) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t) \tag{49}$$

for all t in I.

This means we just have to find one particular solution $\mathbf{x}_p$.

Then the general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$,

where $\mathbf{x}_c$ is the general homogeneous solution.

Two methods to find a particular solution:

- Method of undetermined coefficients
- Variation of parameters

Example: method of undetermined coefficients

Example 1 Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}}_{A}\mathbf{x} + \underbrace{\begin{bmatrix} 3 \\ 2t \end{bmatrix}}_{f(t)}. \tag{3}$$

Guess

$$\vec{x}_p(t) = \begin{bmatrix} a_1t+b_1 \\ a_2t+b_2 \end{bmatrix}.$$

Sub into (3):

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} a_1t+b_1 \\ a_2t+b_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}.$$

$Wt + Y$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_1t + 3b_1 + 2a_2t + 2b_2 + 3 \\ 7a_1t + 7b_1 + 5a_2t + 5b_2 + 2t \end{bmatrix}.$$

$$\boxed{\begin{aligned} a_1 &= 3b_1 + 2b_2 + 3 \\ a_2 &= 7b_1 + 5b_2 \end{aligned}}$$

Then solve this.

$b_1 = 17, \quad b_2 = -25.$

$$\boxed{\begin{aligned} 0 &= 3a_1 + 2a_2 \\ 0 &= 7a_1 + 5a_2 + 2 \end{aligned}}$$

Solve this first.

$a_1 = 4$

$a_2 = -6$

So $x_p = \begin{bmatrix} 4t+17 \\ -6t-25 \end{bmatrix}.$

General soln is

$$\vec{x} = \underbrace{x_c}_{\text{general homogeneous soln.}} + x_p$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

Principle of superposition:

THEOREM 4 Solutions of Nonhomogeneous Systems

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$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t) \qquad (49)$$

for all t in I.

This means we just have to find one particular solution $\mathbf{x}_p$.
Then the general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$
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Example: method of undetermined coefficients

Example 1 Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}. \qquad (3)$$

$\Phi(t) = [x_1(t) \quad x_2(t), \; \cdots \; x_n(t)]$.

To solve

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

consists of the linearly independent solns?

We assume that we have already found the fundamental matrix to the homogeneous equation, $\mathbf{\Phi}(t)$

$x' = P(t)x$

$\Phi' = P(t)\Phi$

Therefore the general homogeneous solution is $\mathbf{x}_c = \mathbf{\Phi}(t)\mathbf{c}$

unknown vector function.

Then we guess $\mathbf{x}_p = \mathbf{\Phi}(t)\mathbf{u}(t)$, and see what constraints on $\mathbf{u}(t)$ come up.

(Then solve for u, hence get x_p)

$x_p' = \Phi'(t)u(t) + \Phi(t)u'(t)$

So plugging in:

$\cancel{\Phi'(t)u(t)} + \Phi(t)u'(t) = P(t)\Phi(t)u(t) + f(t)$

$= \cancel{\Phi'(t)u(t)} + f(t)$

$\Rightarrow \Phi(t)u'(t) = f(t)$

$\Rightarrow u'(t) = \Phi(t)^{-1} f(t)$.

$\Rightarrow u(t) = \int \Phi(t)^{-1} f(t)\, dt$.

$\Rightarrow x_p(t) = \Phi(t) \int \Phi(t)^{-1} f(t)\, dt$.

THEOREM 1 Variation of Parameters

If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval where $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)\, dt. \tag{22}$$

$$\int te^{-7t}\,dt = Ate^{-7t} + Be^{-7t} = \frac{-1}{7}te^{-7t} + \frac{1}{49}e^{-7t}$$

$$\left(Ate^{-7t} + Be^{-7t}\right)' = \underline{A}e^{-7t} + \underline{A}t(-7)e^{-7t} + \underline{-7}Be^{-7t}$$

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Example 4 Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x} - \overbrace{\begin{bmatrix} 15 \\ 4 \end{bmatrix} te^{-2t}}^{f(t)}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}. \tag{30}$$

Note: in previous lectures we found the two solutions to the homogeneous equation,

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$

$$x' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} x$$

* Fundamental matrix:

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

* $$\Phi(t)^{-1} = \frac{1}{\det \Phi(t)} \begin{bmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= \frac{1}{7e^{3t}} \begin{bmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix}$$

So

$$x_p(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \int \frac{1}{7e^{3t}} \begin{bmatrix} e^{5t} & -2e^{5t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -15te^{-2t} \\ -4te^{-2t} \end{bmatrix} dt$$

$$x_p(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \int \frac{1}{7e^{3t}} \begin{bmatrix} -15te^{3t} + 8te^{3t} \\ -45te^{-4t} - 4te^{-4t} \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \int \begin{bmatrix} -15t + 8t \\ -45te^{-7t} - 4te^{-7t} \end{bmatrix} dt$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \int \begin{bmatrix} -7t \\ -49te^{-7t} \end{bmatrix} dt$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{7t^2}{2} \\ 7te^{-7t} + e^{-7t} \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} e^{-2t}\left(-\frac{7}{2}\right)t^2 + 14te^{-2t} + 2e^{-2t} \\ \frac{21t^2}{2}e^{-2t} + 7te^{-2t} + e^{-2t} \end{bmatrix} = \frac{1}{7}e^{-2t}\begin{bmatrix} -\frac{7t^2}{2} + 14t + 2 \\ \frac{21t^2}{2} + 7t + 1 \end{bmatrix}$$

General solution is

$$x = c_1\begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} + c_2\begin{bmatrix} 2e^{5t} \\ e^{5t} \end{bmatrix} + \frac{1}{7}e^{-2t}\begin{bmatrix} -\frac{7t^2}{2} + 14t + 2 \\ \frac{21t^2}{2} + 7t + 1 \end{bmatrix}$$

So

$$x(0) = c_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + 2c_2 + 2/7 = 7 \\ -3c_1 + c_2 + 5/7 = 3 \end{bmatrix}.$$

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