MAT303: Calc IV with applications

Lecture 23 - April 28 2021

Last time: Matrix exponentials

- Review matrix inverses
- Fundamental matrix solutions
  - Solve for all initial conditions ‘simultaneously’.
- Matrix Exponentials as matrix solutions
  - Especially easy to compute when the matrix is nilpotent.

$A^n = 0$ eventually.

Today:

- More examples of matrix exponentials.
  - How to compute them if the matrix is not nilpotent?

Using matrix exponential to solve DEs

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Find $e^{\mathbf{A}t}$
3. Solution is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$ where $\mathbf{x}_0$ is the initial condition.

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots.$$

$$\frac{d}{dt}e^{At} = 0 + A + A^2 t + A^3\frac{3t^2}{3!} + \cdots = A e^{At}.$$

Fundamental Solutions as Matrix Exponentials

Example:

Find solution to IVP $\vec{x}' = \vec{A}\vec{x}$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\vec{x}(0) = \vec{x}_0$.

$$e^{\vec{A}t} = I + \vec{A}t + \vec{A}^2\frac{t^2}{2!} + \cdots$$

$$\vec{A}^2 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{A}^3 = \vec{A}^2\vec{A} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\vec{A}^k = 0$ for $k \geq 3$.

So

$$e^{\vec{A}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}t + \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{t^2}{2}$$

$$= \begin{pmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}$$

Check that the method works (all the columns are solutions to the DE):

3rd column: $\vec{x} = \begin{pmatrix} 4t+9t^2 \\ 6t \\ 1 \end{pmatrix}$.

$$\vec{x}' = \begin{pmatrix} 4+18t \\ 6 \\ 0 \end{pmatrix}$$

$$\vec{A}\vec{x} = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4t+9t^2 \\ 6t \\ 1 \end{pmatrix} = \begin{pmatrix} 18t+4 \\ 6 \\ 0 \end{pmatrix}.$$

If $A^n = 0$ for some n, the matrix is said to be nilpotent.
We just saw that it is easy to compute e^{At} when A is nilpotent.

12

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

We've just seen that computing $e^{\mathbf{A}t}$ is useful for solving systems of DEs.

$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{4t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}$.

Here are some facts that help us compute $e^{\mathbf{A}t}$.

Definition of matrix exponential:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}\,t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots.$$

$A^n = 0$

Nilpotent A : Just use the definition, it will be a finite sum because $\cancel{\mathbf{A}}^n$ eventually.

Diagonal A: if A is diagonal, $A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ (zero off the diagonal)

$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$

Why? $A^n = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^n = \begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_n^n \end{bmatrix}$

So $e^{At} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & \\ & \ddots & \\ & & \lambda_n t \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2 t^2}{2!} & & \\ & \ddots & \\ & & \frac{\lambda_n^2 t^2}{2!} \end{bmatrix} + \cdots$

Commutativity: If $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.

Just like $e^{x+y} = e^x e^y$.

(See problem 31 in Ch 5.6).

Exponential of zero matrix: $e^{\mathbf{0}} = \mathbf{I}$.

$\vec{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$e^{\vec{0}} = I = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

Inverse of exponential: $\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}}$

Why?

$e^{A+(-A)} = e^0 = I$

So $e^A \cdot e^{-A} = I \Rightarrow e^{-A} = (e^A)^{-1}$.

~~**Diagonal matrix always commute with all other matrices:**~~

Correction May 4:

This is not true.

What I meant to say is that "λI commutes with all other matrices".

E.g. $\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ is diagonal but not of the form λI

Suppose $\det(A-\lambda I) = (\lambda-2)^2$.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

Let's compute e^{At}.

$$A = \underbrace{\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}}_{2I} + \underbrace{\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}}_{B}.$$

diagonal.

So

$$e^{At} = e^{(2I+B)t} = e^{2It + Bt}$$

$$= e^{2It} e^{Bt}$$

$$= \begin{pmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{2t} \end{pmatrix}\left(I + Bt + \frac{B^2t^2}{2!} + \cdots \right).$$

B is nilpotent:

$$B^2 = \begin{pmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B^3 = 0.$$

So.

$$e^{At} = \begin{pmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{2t} \end{pmatrix}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}t + \begin{pmatrix} 0 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}t^2 \right)$$

$$= e^{2t}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}.$$

Fact: whenever a matrix $\mathbf{A}$ only has one eigenvalue λ, we can always write $\mathbf{A} = \lambda\mathbf{I} + (\mathbf{A} - \lambda\mathbf{I})$ and the first term is always ~~diagonal~~ λI and the second term is always nilpotent.

Thus, when a matrix only has one eigenvalue, we can easily compute $e^{\mathbf{A}t}$ as in this example.

Recall (from last lecture)

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

Example 6 Use an exponential matrix to solve the initial value problem

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 19 \\ 29 \\ 39 \end{bmatrix}. \tag{29}$$

From last slide,

$$e^{At} = e^{2t}\begin{pmatrix} 1 & 3t & 4t \; 19t \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}.$$

So soln is

$$x(t) = e^{2t}\begin{pmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 19 \\ 29 \\ 39 \end{pmatrix}$$

$$x(t) = e^{2t}\begin{pmatrix} 19 + 87t + 156t + 351t^2 \\ 29 + 234t \\ 39 \end{pmatrix}.$$

$(A+3I)^2 v_2 = 0.$

Another example, same one we did last lecture using a different method:

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v_2 = 0$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$

$v_a =$ anything

Example 3 Find a general solution of the system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}}_{A}\mathbf{x}. \qquad (20)$$

$\begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}$

Last lecture:

Characteristic polynomial is $\lambda = -3.$

$\det(A-\lambda I) = (\lambda-4)^2$

Only 1 eigenvalue of A is $\lambda = 4$.

Only eigenvector is $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So eigenvalue is defective with multiplicity 2.

To find general solution,

such that $(A-\lambda I)v_2 \neq 0$.

$* \; A-\lambda I = \begin{pmatrix} 1-4 & -3 \\ 3 & 7-4 \end{pmatrix} = \overbrace{\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}}^{A-4I}$

$(A-\lambda I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So $(A-\lambda I)^2 v_2 = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v_2 = 0.$ (1)

$\Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$

Many solutions, take $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (make sure).

$\left(v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \; v_2 = \begin{pmatrix} 7 \\ 4 \end{pmatrix}, \; \cancel{v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}\right).$

$\cancel{v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$v_1 = (A-\lambda I)v_2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

② The 2 linearly dependent sols corresponding to $\lambda = 4$

$\vec{x}_1(t) = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t}, \quad \vec{x}_2(t) = (v_1 t + v_2)e^{4t}$

* Unlike in (1), you will not always get 0 constraints. But you will get underdetermined system.

$$\vec{x}_2 = \left(\begin{pmatrix} -3 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) e^{4t}$$

$$\Phi(t) = \begin{pmatrix} -3e^{4t} & (-3t+1)e^{4t} \\ 3e^{4t} & 3te^{4t} \end{pmatrix}$$

We see that there is only one eigenvalue $\lambda = 4$, so

$B = A - 4I = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}$ is nilpotent.

$B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

So

$$e^{At} = e^{4It + (A-4I)t} = e^{4It + Bt}$$

$$= e^{4It} e^{Bt}$$

$$= \begin{pmatrix} e^{4t} & \\ & e^{4t} \end{pmatrix}\left(I + Bt + \frac{B^2 t^2}{2!} + \cdots\right)$$

$$= \begin{pmatrix} e^{4t} & \\ & e^{4t} \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} t\right)$$

$$= \begin{pmatrix} e^{4t} & \\ & e^{4t} \end{pmatrix}\begin{pmatrix} 1-3t & -3t \\ 3t & 1+3t \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} \boxed{(1-3t)e^{4t}} & -3t\, e^{4t} \\ 3t\, e^{4t} & (1+3t)e^{4t} \end{pmatrix}$$

So soln to any IVP is

$$x(t) = \begin{pmatrix} (1-3t)e^{4t} & -3t\, e^{4t} \\ 3t\, e^{4t} & (1+3t)e^{4t} \end{pmatrix} \vec{x}_0 .$$

We have, from last lecture:

$$\underline{\Phi}(t) = [\vec{x}_1(t) \;\cdots\; \vec{x}_n(t)].$$

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \boxed{\mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = \boxed{e^{\mathbf{A}t}}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

~~Therefore:~~

Ex:

$$\underline{\Phi}(t) = \begin{pmatrix} -3e^{4t} & (-3t+1)e^{4t} \\ 3e^{4t} & 3te^{4t} \end{pmatrix} \quad \underline{\Phi}(0) = \begin{pmatrix} -3 & 1 \\ 3 & 0 \end{pmatrix} \Rightarrow \underline{\Phi}(0)^{-1} = \frac{-1}{3}\begin{pmatrix} 0 & -3 \\ -1 & -3 \end{pmatrix}.$$

$$\underline{\Phi}(t)\underline{\Phi}(0)^{-1} = -\frac{1}{3}\begin{pmatrix} \boxed{(1-3t)e^{4t}} & - \\ - & - \end{pmatrix}$$

Therefore:

$$\underline{\Phi}(t)\underline{\Phi}(0)^{-1} = e^{At}.$$

Summary: we have seen that matrix exponentials can be used to solve DEs.

However, we only know how to compute matrix exponentials for some matrices.

We can go the other direction and solve DEs to find matrix exponentials.

Today:

- More examples of matrix exponentials.
  - How to compute them if the matrix is not nilpotent?
  - We see that it is easy as long as A=diagonal + nilpotent
  - Actually whenever A only has one eigenvalue, we can write it as ~~diagonal~~ λI + nilpotent.
- We see that we can use DE solutions to find matrix exponentials.

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions if applicable.

Using matrix exponential to solve DEs

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Find $e^{\mathbf{A}t}$
3. Solution is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$