MAT303: Calc IV with applications

Lecture 22 - April 26 2021

Recently:

- Eigenvalue method for $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - Still need to finish off case of defective eigenvalues (missing solutions).

Today: Ch 5.6.

- Review matrix inverses
- Fundamental matrix solutions
  - Solve for all initial conditions ‘simultaneously’.
- Matrix Exponentials as fundamental matrix solutions

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
   1. Some complications if the eigenvalue is defective (not enough eigenvectors)
5. Write down the solutions, solve for initial conditions if applicable.

Finding more solutions when there are defective eigenvalues

Let's start with the multiplicity $k = 2$ case, it's the simplest.

Situation:

- We are trying to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- The matrix $\mathbf{A}$ has an eigenvalue λ of multiplicity 2 (repeated root)
- The eigenvalue λ is defective (only 1 linearly independent eigenvector $\mathbf{v}_1$ instead of 2).
- So we only have one solution, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$.
- Need to find another.

Solution: guess $\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ where $\mathbf{v}_2$ is ~~unknown~~.

We find that the constraint on $\mathbf{v}_2$ is $(\mathbf{A} - \lambda I)^2 \mathbf{v}_2 = 0$

Note: once we find $\mathbf{v}_2$ then $\mathbf{v}_1 = (\mathbf{A} - \lambda I)\mathbf{v}_2$.

ALGORITHM Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{v}_2 = \mathbf{0} \tag{16}$$

such that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{17}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{18}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \tag{19}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 3 Find a general solution of the system

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}}_{A}\mathbf{x}. \qquad (20)$$

Last lecture:

Characteristic polynomial is

$$\det(A - \lambda I) = (\lambda - 4)^2$$

Only 1 eigenvalue of A is $\lambda = 4$.

Only eigenvector is $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

So eigenvalue is defective with multiplicity 2.

To find general solution,

1. Solve $(A - \lambda I)^2 v_2 = 0$ such that $(A - \lambda I) v_2 \neq 0$.

* $A - \lambda I = \begin{pmatrix} 1-4 & -3 \\ 3 & 7-4 \end{pmatrix} = \overbrace{\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}}^{A-4I}$

$(A - \lambda I)^2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So $(A - \lambda I)^2 v_2 = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} v_2 = 0$. (1)

$\Leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_a \\ v_b \end{pmatrix} = 0$

Many solutions, take $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (make sure).

$\left(v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix},\ v_2 = \begin{pmatrix} 7 \\ 4 \end{pmatrix},\ \cancel{v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \right).$

$\cancel{v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$v_1 = (A - \lambda I) v_2 = \begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

② The 2 linearly dependent sols corresponding to $\lambda = 4$

$\vec{x}_1(t) = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^{4t}, \quad \vec{x}_2(t) = (v_1 t + v_2) e^{4t}$

* Unlike in (1), you will not always get 0 constraints. But you will get underdetermined system.

Ch 5.6 Exponential Matrices and Fundamental Matrix Solutions

let $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ identity matrix.

$$IA = A = AI$$

let A be a square matrix. Inverse to A is the matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

Example:

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}.$$

Check:

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 1+6 & -2+2 \\ -3+3 & 6+1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

Formula for 2×2 inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Useful for solving linear systems:

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{cases} c_1 + 2c_2 = 1 \\ -3c_1 + c_2 = 1 \end{cases}$$

Multiply both sides by $\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1}$ (on the left)

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow I \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$= \frac{1}{7} \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$x(0) = 1, \quad y(0) = 1.$

$\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix}.$

We know how to find the general solution now: $\begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{5t} \\ -3c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}$

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions.

$\vec{x}_1 = \begin{bmatrix} 1e^{-2t} \\ -3e^{-2t} \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} 2e^{5t} \\ 1e^{5t} \end{bmatrix}.$

general soln:

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

$= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Solve for IC's:

$\vec{x}(0) = \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

So $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ 4 \end{bmatrix}.$

previous slide

So soln to IVP is

$\vec{x} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \cdot \begin{bmatrix} -1/7 \\ 4/7 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$(x(t), y(t))$

$x(0) = 3, \quad y(0) = 2.$

$$\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix}.$$

We know how to find the general solution now:

$$\begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + 2c_2 e^{5t} \\ -3c_1 e^{-2t} + c_2 e^{5t} \end{bmatrix}$$

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions.

Insight: The IVP solution can be written as a product of a matrix and a vector.

general soln:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$
$$= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}.$

Solve for IC's:

$$\vec{x}(0) = \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

So $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 \\ 11 \end{bmatrix}.$

previous slide

So soln to IVP is

$$\vec{x} = \underline{\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}} \cdot \begin{bmatrix} -1/7 \\ 11/7 \end{bmatrix}.$$

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$y(1)$)

$x(0) = u_1, \quad y(0) = u_2.$

$$\overset{\Phi}{\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}} = \begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix}.$$

We know how to find the general solution now:

$$\begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1e^{-2t} + 2c_2e^{5t} \\ -3c_1e^{-2t} + c_2e^{5t} \end{bmatrix}$$

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions.

Insight: Just use the matrix inverse instead of solving for the coefficients.

general soln:

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$
$$= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\frac{1}{7}[$

Solve for IC's:

$$\vec{x}(0) = \begin{bmatrix} \vec{x}_1(0) & \vec{x}_2(0) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

So $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} =$

So soln to IVP is

$$\vec{x} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Definition: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be *linearly independent* solutions the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Let $\mathbf{\Phi}$ be the matrix formed by taking $\mathbf{x}_i$ as the columns.

Then $\mathbf{\Phi}$ is said to be a fundamental matrix for the system.

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

The previous example, summarized with this new vocabulary:

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$$\Phi(t) = [\vec{x}_1 \ \ \vec{x}_2] = \begin{bmatrix} 1e^{-2t} & 2e^{5t} \\ -3e^{-2t} & 1e^{5t} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

So soln to IVP is

$$\vec{x}(t) = [\vec{x}_1 \ \ \vec{x}_2] \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1} \vec{x}_0.$$

(First example: $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2nd: $\vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

3rd $\vec{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$)

Takeaway: We can solve the system for *all* initial conditions, all at once. Just compute $\Phi(t)\Phi(0)^{-1}$.

It turns out that there's another, conceptually cleaner way to view fundamental solutions.

Also, this can sometimes lead to a much quicker computation.

It is inspired by the following fact:

The solution to $x' = ax$ is $x(t) = e^{at}x(0)$.

Explanation:
Soln is $x(t) = e^{at} C$

Plugging in $t=0$,
get $x(0) = C$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

This doesn't make sense yet, because what does $e^{\mathbf{A}t}$ mean???

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Similarly, we define

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots.$$

Looks complicated because it's an infinite sum, but there are some tricks that can help us.

Example:

Find solution to IVP $\vec{x}' = \vec{A}\vec{x}$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\vec{x}(0) = x_0$.

$$e^{\vec{A}t} = I + \vec{A}t + \vec{A}^2 \frac{t^2}{2!} + \cdots$$

$$\vec{A}^2 = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{A}^3 = \vec{A}^2\vec{A} = \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\vec{A}^k = 0$ for $k \geq 3$.

So

$$e^{\vec{A}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{t^2}{2}$$

$$= \begin{pmatrix} 1 & 3t & 4t+9t^2 \\ 0 & 1 & 6t \\ 0 & 0 & 1 \end{pmatrix}$$

Check that the method works (all the columns are solutions to the DE):

3rd column: $\vec{x} = \begin{pmatrix} 4t+9t^2 \\ 6t \\ 1 \end{pmatrix}$.

$$\vec{x}' = \begin{pmatrix} 4+18t \\ 6 \\ 0 \end{pmatrix}$$

$$\vec{A}\vec{x} = \begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4t+9t^2 \\ 6t \\ 1 \end{pmatrix} = \begin{pmatrix} 18t+4 \\ 6 \\ 0 \end{pmatrix}.$$

If $A^n = 0$ for some n, the matrix is said to be nilpotent.
We just saw that it is easy to compute e^{At} when A is nilpotent.

Recently:

- Eigenvalue method for $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Today:

- Review matrix inverses
- Fundamental matrix solutions
  - Solve for all initial conditions ‘simultaneously’.
- Matrix Exponentials as matrix solutions
  - Especially easy to compute when the matrix is nilpotent.

Next time.

~~Today~~:

- More examples of matrix exponentials.
  - How to compute them if the matrix is not nilpotent?

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions if applicable.

Using matrix exponential to solve DEs

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Find $e^{\mathbf{A}t}$
3. Solution is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$