MAT303: Calc IV with applications

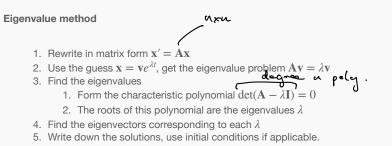
Lecture 21 - April 21 2021

Last time:

- Linear independence of solutions (Finish Ch 5.1)
- Eigenvalue method (Ch 5.2)
 - Distinct real eigenvalues

Today:

- · Eigenvalue method
 - Distinct complex eigenvalues (Ch 5.2)
 - Repeated eigenvalues (Ch 5.3)



Recall: Euler's identity

 $e^{ix} = \cos(x) + i\sin(x)$

Recall: Complex roots of polynomials appear in conjugate pairs

If p + qi is a root of a polynomial with real coefficients, then p - qi is also a root.

Recall: Superposition principle C. R. + C2R2. If $\mathbf{x}_1, \mathbf{x}_2$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then so is **equation**. Recall: Complex conjugation If z = p + qi then $\overline{z} = p - qi$. Recall: if x-achi Re(x) = a $\int m(x) = b.$ Note: $f(x+\overline{x}) = f(a+bi+a-bi) = a$. Another fact: complex eigenvectors appear in pairs

If v is the eigenvector of A corresponding to eigenvalue λ Then v is the eigenvector of A corresponding to eigenvalue $\overline{\lambda}$

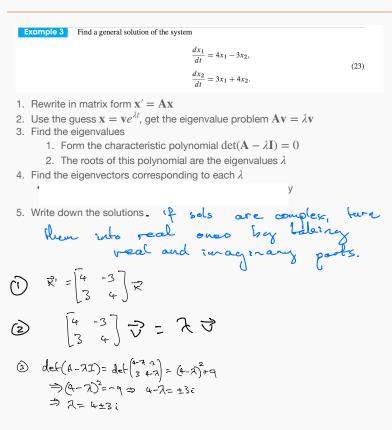
In other words,
If
$$Av = \lambda v$$

then $A\overline{v} = \overline{\lambda}\overline{v}$
Consequence:
If \overline{X} is a complex solution to the
DE $\overline{X}^{i} = \overline{A} \cdot \overline{X}$,
then $Re[\overline{X}]$ and $Im(\overline{X})$ are
real solutions.
Shy?: \overline{X} and \overline{X} are solutions,
 $So \frac{1}{2}\overline{X} + \frac{1}{2}\overline{X}$ is a solution.

We can use these facts deal with the case when there are complex eigenvalues.

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Complex eigenvalues in the eigenvalue method



Find eigenvectors:

$$\lambda = 4-3i:$$

$$\begin{bmatrix} x - 5 \\ 3 + 6 \end{bmatrix} \begin{bmatrix} y \\ -3 \end{bmatrix} \begin{bmatrix} y$$

Example 3 Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2,$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2.$$
(23)

- 1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- 2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$
- 3. Find the eigenvalues
 - 1. Form the characteristic polynomial det($\mathbf{A} \lambda \mathbf{I}$) = 0
 - 2. The roots of this polynomial are the eigenvalues λ
- 4. Find the eigenvectors corresponding to each λ
 - If complex, pair up conjugates and use Euler's identity to get real solutions
- 5. Write down the solutions

Recall: Euler's identity

 $e^{ix} = \cos(x) + i\sin(x)$

Recall: Complex roots of polynomials appear in conjugate pairs

If p + qi is a root of a polynomial with real coefficients, then p - qi is also a root.

Recall: Superposition principle

If x_1, x_2 are solutions to x' = Ax, then so is $x_1 + x_2$.

Recall: Complex conjugation

If z = p + qi then $\overline{z} = p - qi$.

Another fact: complex eigenvectors appear in pairs

If v is the eigenvector of A corresponding to eigenvalue λ . Then v is the eigenvector of A corresponding to eigenvalue $\bar{\lambda}$

Last time: we saw that

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \text{ and } \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a$, $\mathbf{x}'(0) = b$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be *n* linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval *I*, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on *I*, then there exist numbers c_1 , c_2, \dots, c_n such that

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \dots + c_n \mathbf{x}_n(t)$$
(35)

for all t in I.

Takeaway: for a $n \times n$ linear system, once we find n linearly independent solutions, we have essentially found them 'all'.

Repeated eigenvalues (Ch 5.5)

Repeated eigenvalues in the eigenvalue method

Example 1 Find a general solution of the system $\begin{array}{c} \mathbf{x}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \mathbf{x}. \end{array}$ (5) O AJ=27 3 def (A-ZI) = def (-6 -1-2 0) 6 4 3-2) = $(5 - R)(3 - R)^2$ So eign are R = 5, 3. @ Finding eigenvectors: I げ えっち、 マー「---」 X- X-3: $\begin{bmatrix} q & 4 & 0 \\ -1 & 0 \\ r & 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 3 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\begin{cases}
q_{u_1} + q_{v_2} = 3u_1 \\
= 6u_1 - v_2 = 3u_2 \\
6u_1 + q_{u_2} + 3u_3 = 3v_3.
\end{cases}$

 $\begin{cases} G_{U_1} + u_{V_2} = 0 \\ -G_{U_1} - 4V_2 = 0 \\ G_{U_1} + u_{U_2} = 0 \\ -G_{U_1} + u_{U_2} = 0 \\ -G_{U_1} + u_{U_2} = 0 \end{cases}$ Only constraint: $V_2 = -\frac{3}{2}U_1$ So all the solo are of flue form. $\frac{V_2}{V_2} = -\frac{3}{2}U_1$ Just take (;) and (-3/2). Choose que 3 General sola is make some that the results are undependent. $C_1[-i]e^{5t} + C_2[0]e^{5t} + C_3[-3c_2]$

We are always be looking for n linearly independent eigenvectors, to make sure we have found all solutions.

If an eigenvalue of multiplicity k has k linearly independent eigenvectors, it is said to be **complete.**

3 is a complete sigenvalue.

However, when there are repeated roots, there are sometimes there are not enough linearly independent eigenvectors...

Defective eigenvalues

You should always be looking for n linearly independent eigenvectors.

However, sometimes there are not enough linearly independent eigenvectors...

(8)

The following matrix only has one eigenvector.

Example 2 The matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$
Charachershie polynomial:
$det(4-\lambda I) = det\left(\begin{array}{c} (-\lambda - 3) \\ 3 \\ -\lambda \end{array}\right)$
= (1-x)(7-x) + 9
$= \lambda^{2} + 7 - 7\lambda - 7 - 19$
$= \chi^2 - 8\chi + 16$
$= (x - \theta)^2$
Eig is 4.
Find eigvect:
7-4:
$\begin{bmatrix} r & -3\\ 3 & 7 \end{bmatrix} \begin{bmatrix} v_1\\ v_2 \end{bmatrix} = \begin{bmatrix} v_1\\ 4\\ v_2 \end{bmatrix}$

$$\Rightarrow \begin{cases} V_1 - 3v_2 = 4v_1 \\ 3v_1 + 7v_2 = 4v_2 \end{cases}$$

$$\Rightarrow \begin{cases} -3v_1 - 3v_2 = 6 \\ 3v_1 + 3v_2 = 0. \end{cases} \qquad redundant$$

$$\Rightarrow \qquad V_2 = -V_1$$
So sols one
$$\binom{V_1}{-V_4} = U_1\binom{-1}{-1}.$$
Every eigenvector is a multiple
$$\Rightarrow \binom{(V_1)}{-1} = \binom{-1}{-1}.$$

$$= 4 \text{ is not complete.}$$

$$4 \text{ is a defective eigenvalue.}$$

Situation:

• We are trying to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$

- The matrix A has an eigenvalue λ of multiplicity 2 (repeated root)
- The eigenvalue λ is defective (only 1 linearity independent eigenvector v_1 instead of 2).
- So we only have one solution, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$.
- Need to find another.

Solution: guess $\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ where \mathbf{v}_2 is unknown.

$$v_{1}\left(e^{2t} + t^{2}e^{2t}\right) + v_{2}^{2}e^{2t} = A\left(v_{1} + t^{2}e^{2t}\right)$$

$$v_{1}\left(e^{2t} + t^{2}e^{2t}\right) + v_{2}^{2}e^{2t} = A\left(v_{1} + t^{2}e^{2t}\right)$$

$$v_{1}e^{2t} + 2v_{1}te^{2t} + v_{2}^{2}e^{2t} = Av_{2} + Av_{2}e^{2t}$$

$$\Rightarrow \int v_{1} + v_{2}^{2} = Av_{2} \Rightarrow (A - 2t)v_{2} = v_{1} \quad (1)$$

$$Av_{1} = 2v_{1} \quad \Rightarrow (A - 2t)v_{1} = \mathbf{0} \quad (2)$$

Finding more solutions when there are defective eigenvalues

We find that the constraint on \mathbf{v}_2 is $(\mathbf{A} - \lambda I)^2 \mathbf{v}_2 = 0$

Note: once we find \mathbf{v}_2 then $\mathbf{v}_1 = (\mathbf{A} - \lambda I)\mathbf{v}_2$.

Materphy (1) by
$$(A - \pi I)$$

 $\Rightarrow (A - \pi I)^2 v_2 = (A - \pi I) v_1 = 0$

ALGORITHM Defective Multiplicity 2 Eigenvalues **1.** First find a nonzero solution \mathbf{v}_2 of the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0} \tag{16}$$

such that

$$\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{17}$$

is nonzero, and therefore is an eigenvector \mathbf{v}_1 associated with λ . 2. Then form the two independent solutions

x

(A

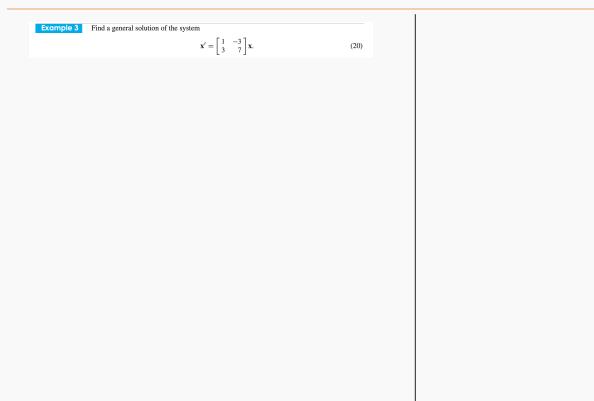
$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{18}$$

and

$$\mathbf{v}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t} \tag{19}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ .

Example of the algorithm



Today:

- Eigenvalue method
 - Distinct complex eigenvalues (Ch 5.2)
 - Just use Euler's formula + superposition
 - Repeated eigenvalues (Ch 5.3)
 - If the eigenvalues are defective, must look for generalized eigenvectors
 - We only did multiplicity k = 2, but the same thing works for higher multiplicity.