MAT303: Calc IV with applications

Lecture 21 - April 21 2021

Last time:

- Linear independence of solutions (Finish Ch 5.1)
- Eigenvalue method (Ch 5.2)
  - Distinct real eigenvalues

Today:

- Eigenvalue method
  - Distinct complex eigenvalues (Ch 5.2)
  - Repeated eigenvalues (Ch 5.3)

Eigenvalue method

n×n

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\overbrace{\det(\mathbf{A} - \lambda\mathbf{I})}^{\text{degree } n \text{ poly.}} = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, use initial conditions if applicable.

Recall: **Euler's identity**

$$e^{ix} = \cos(x) + i\sin(x)$$

Recall: **Complex roots of polynomials appear in conjugate pairs**

If $p + qi$ is a root of a polynomial with real coefficients, then $p - qi$ is also a root.

Recall: **Superposition principle**

If $\mathbf{x}_1, \mathbf{x}_2$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then so is ~~$\mathbf{x}_1 + \mathbf{x}_2$~~. $c_1\vec{x}_1 + c_2\vec{x}_2$.

Recall: **Complex conjugation**

If $z = p + qi$ then $\bar{z} = p - qi$.

Recall: if

$x = a + bi$

$\mathrm{Re}(x) = a$

$\mathrm{Im}(x) = b.$

Note: $\frac{1}{2}(x + \bar{x}) = \frac{1}{2}(a + bi + a - bi) = a.$

Another fact: **complex eigenvectors appear in pairs**

If $\mathbf{v}$ is the eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ

Then $\bar{\mathbf{v}}$ is the eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\bar{\lambda}$

In other words,

If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

then $\mathbf{A}\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$

E.g. if $\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\overline{\vec{v}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$

Consequence: if $\vec{x}$ is a complex solution to the DE $\vec{x}' = \vec{A}\vec{x}$, then $\mathrm{Re}[\vec{x}]$ and $\mathrm{Im}(\vec{x})$ are real solutions.

Why? $\vec{x}$ and $\overline{\vec{x}}$ are solutions, so $\frac{1}{2}\vec{x} + \frac{1}{2}\overline{\vec{x}}$ is a solution. But this is $\mathrm{Re}(\vec{x})$.

We can use these facts deal with the case when there are complex eigenvalues.

Example 3 Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2, \qquad \frac{dx_2}{dt} = 3x_1 + 4x_2. \tag{23}$$

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions. if sols are complex, turn them into real ones by taking real and imaginary parts.

(1) $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \vec{x}$

(2) $\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \vec{v} = \lambda \vec{v}$

(3) $\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 + 9$

$\Rightarrow (4-\lambda)^2 = -9 \Rightarrow 4-\lambda = \pm 3i$

$\Rightarrow \lambda = 4 \pm 3i$

(4) Find eigenvectors:

$\lambda = 4-3i$:

$\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (4-3i)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Real → imaginary

$\begin{cases} 4v_1 - 3v_2 = (4-3i)v_1 \\ 3v_1 + 4v_2 = (4-3i)v_2 \end{cases}$

$\Rightarrow \begin{cases} 3iv_1 - 3v_2 = 0 \\ 3v_1 + 3iv_2 = 0. \end{cases} \Rightarrow \begin{cases} iv_1 - v_2 = 0 \\ v_1 + iv_2 = 0 \end{cases}$

Second eqn is redundant, the only constraint is $v_2 = iv_1$

So eigenvector is $\begin{bmatrix} v_1 \\ iv_1 \end{bmatrix}$.

Just take $v_1 = 1$, so $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

(if $v_2 = 2$, get $\begin{bmatrix} 2 \\ 2i \end{bmatrix}$). doesn't matter.

$\lambda = 4+3i$

Corresponding eigenvector is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$. because fact on prev slide.

(5) $x = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{4t} e^{-3it}$

$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{4t}(\cos 3t - i\sin(3t))$

$= e^{4t}\begin{bmatrix} \cos 3t - i\sin 3t \\ i\cos 3t + \sin(3t) \end{bmatrix}$.

is a soln.

So $x_1 = \mathrm{Re}(x) = e^{4t}\begin{bmatrix} \cos 3t \\ \sin(3t) \end{bmatrix}$

$x_2 = \mathrm{Im}(x) = e^{4t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}$

General soln is $c_1 e^{4t}\begin{bmatrix} \cos 3t \\ \sin(3t) \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}$.

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2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
    - If complex, pair up conjugates and use Euler's identity to get real solutions
5. Write down the solutions

Recall: **Euler's identity**

$e^{ix} = \cos(x) + i\sin(x)$

Recall: **Complex roots of polynomials appear in conjugate pairs**

If $p + qi$ is a root of a polynomial with real coefficients, then $p - qi$ is also a root.

Recall: **Superposition principle**

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Recall: **Complex conjugation**

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Last time: we saw that

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a, \quad \mathbf{x}'(0) = b$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers c_1, $c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{35}$$

for all t in I.

Takeaway: for a $n \times n$ linear system, once we find n linearly independent solutions, we have essentially found them ‘all’.

Repeated eigenvalues (Ch 5.5)

Example 1 Find a general solution of the system

① $$\mathbf{x}' = \underbrace{\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}}_{A}\mathbf{x}. \qquad (5)$$

② $A\vec{v} = \lambda\vec{v}$

③ $\det(A-\lambda I) = \det\begin{pmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{pmatrix}$

$= (5-\lambda)(3-\lambda)^2$

So eigs are $\lambda = 5, 3$.

④ Finding eigenvectors: I

If $\lambda = 5$, $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

If $\lambda = 3$:

$$\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 3\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{cases} 9v_1 + 4v_2 = 3v_1 \\ -6v_1 - v_2 = 3v_2 \\ 6v_1 + 4v_2 + 3v_3 = 3v_3. \end{cases}$$

$$\begin{cases} 6v_1 + 4v_2 = 0 \\ -6v_1 - 4v_2 = 0 \quad \leftarrow \text{redundant.} \\ 6v_1 + 4v_2 = 0 \quad \leftarrow \text{redundant.} \end{cases}$$

Only constraint: $v_2 = -\frac{3}{2}v_1$

So all the sols are of the form. $\begin{pmatrix} v_1 \\ -\frac{3}{2}v_1 \\ v_3 \end{pmatrix}$

Just take $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $(v_1=0, v_3=1)$ and $\begin{pmatrix} 1 \\ -3/2 \\ 0 \end{pmatrix}$ $(v_1=1, v_3=0)$.

Choose anything you want, but make sure that the results are independent.

⑤ General soln is

$$c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{3t} + c_3\begin{bmatrix} 1 \\ -3/2 \\ 0 \end{bmatrix}e^{3t}.$$

We are always be looking for n linearly independent eigenvectors, to make sure we have found all solutions.

If an eigenvalue of multiplicity k has k linearly independent eigenvectors, it is said to be **complete.**

3 is a complete eigenvalue.

However, when there are repeated roots, there are sometimes there are not enough linearly independent eigenvectors…

You should always be looking for n linearly independent eigenvectors.

However, sometimes there are not enough linearly independent eigenvectors…

The following matrix only has one eigenvector.

Example 2 The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \tag{8}$$

Characteristic polynomial:

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{pmatrix}$$
$$= (1-\lambda)(7-\lambda) + 9$$
$$= \lambda^2 + 7 - 7\lambda - \lambda + 9$$
$$= \lambda^2 - 8\lambda + 16$$
$$= (\lambda - 4)^2$$

Eig is 4.

Find eigvect:

$\lambda = 4$:

$$\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4v_1 \\ 4v_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} v_1 - 3v_2 = 4v_1 \\ 3v_1 + 7v_2 = 4v_2 \end{cases}$$

$$\Rightarrow \begin{cases} -3v_1 - 3v_2 = 0 \\ 3v_1 + 3v_2 = 0. \quad \leftarrow \text{redundant} \end{cases}$$

$$\Rightarrow \quad v_2 = -v_1$$

So sols are

$$\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Every eigenvector is a multiple of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

4 is not complete.

4 is a defective eigenvalue.

Let's start with the multiplicity $k = 2$ case, it's the simplest.

Situation:
- We are trying to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- The matrix $\mathbf{A}$ has an eigenvalue λ of multiplicity 2 (repeated root)
- The eigenvalue λ is defective (only 1 linearly independent eigenvector $\mathbf{v}_1$ instead of 2).
- So we only have one solution, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$.
- Need to find another.

Solution: guess $\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ where $\mathbf{v}_2$ is unknown.

Then, plugging into $\vec{x}' = \vec{A}\vec{x}$,

giving

$$v_1 (e^{\lambda t} + t\lambda e^{\lambda t}) + v_2 \lambda e^{\lambda t} = A(v_1 t e^{\lambda t} + v_2 e^{\lambda t})$$

$$v_1 e^{\lambda t} + \lambda v_1 t e^{\lambda t} + v_2 \lambda e^{\lambda t} = A v_1 t e^{\lambda t} + A v_2 e^{\lambda t}$$

$$\Rightarrow \begin{cases} v_1 + v_2 \lambda = A v_2 \Rightarrow (A - \lambda I) v_2 = v_1 & (1) \\ A v_1 = \lambda v_1 . \Rightarrow (A - \lambda I) v_1 = 0 & (2) \end{cases}$$

We find that the constraint on $\mathbf{v}_2$ is $(\mathbf{A} - \lambda I)^2 \mathbf{v}_2 = 0$

Note: once we find $\mathbf{v}_2$ then $\mathbf{v}_1 = (\mathbf{A} - \lambda I)\mathbf{v}_2$.

Multiply (1) by $(A - \lambda I)$

$$\Rightarrow (A - \lambda I)^2 v_2 = (A - \lambda I) v_1 = 0$$

ALGORITHM Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{v}_2 = \mathbf{0} \quad (16)$$

such that

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \quad (17)$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \quad (18)$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \quad (19)$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 3 Find a general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}. \tag{20}$$

Today:

- Eigenvalue method
  - Distinct complex eigenvalues (Ch 5.2)
    - Just use Euler's formula + superposition
  - Repeated eigenvalues (Ch 5.3)
    - If the eigenvalues are defective, must look for generalized eigenvectors
    - We only did multiplicity $k = 2$, but the same thing works for higher multiplicity.