MAT303: Calc IV with applications

Lecture 20 - April 19 2021

Today

Last time:

- Seeing how matrix notation helps us represent systems more compactly
- Basic application of row reduction to solve for coefficients in initial value problems
- Principle of superposition

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned} \tag{27}
$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t).$$

$$P(t) = \begin{bmatrix} p_{11}(t) & & \\ & \ddots & \\ & & p_{nn}(t) \end{bmatrix}$$

Today:

- Linear independence of solutions (Finish Ch 5.1)
- Eigenvalue method (Ch 5.2)

Recall (lecture 11): linear independence of more than two functions:

DEFINITION Linear Dependence of Functions

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{7}$$

on I; that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x in I.

Example: $\{x, x+e^x, e^x\}$ linearly dependent:

$$1x - 1\cdot(x+e^x) + 1\cdot e^x = 0.$$

Definition for vectors is similar:

Independence and General Solutions

Linear independence is defined in the same way for vector-valued functions as for real-valued functions (Section 3.2). The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \tag{32}$$

for all t in I. Otherwise, they are **linearly independent**. Equivalently, they are

Example: $\vec{x}_1 = \begin{bmatrix} 3e^{2t} \\ 2e^{5t} \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} 6e^{2t} \\ 4e^{5t} \end{bmatrix}$

linearly dependent:

$$2\vec{x}_1 - 1\vec{x}_2 = 0.$$

Another way to check linear independence is through the Wronksian, see textbook.

Last time: we saw that

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \text{ and } \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

(superposition principle).

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a, \quad \mathbf{x}'(0) = b$

Compare this to the following single second order equation:

$$y'' - 2y' + y = 0$$

guess $y = e^{rt}$

We can easily find two solutions:

$$y = e^t \text{ and } y = te^t$$

And we can take linear combinations to get new solutions:

$$y = c_1 e^t + c_2 te^t$$

We could choose c_1 and c_2 to match initial conditions $y(0) = a, \quad y'(0) = b$

- We know that once we find two linearly independent solutions, all other solutions are linear combinations.

Last time: we saw that

2x2 matrix

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a, \quad \mathbf{x}'(0) = b$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers c_1, $c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{35}$$

for all t in I.

nxn matrix.

Takeaway: for a $n \times n$ linear system, once we find n linearly independent solutions, we have essentially found them 'all'.

Eigenvalue method (Ch 5.2)

$y = e^{rt} \Rightarrow ay'' + by' + cy = 0$

$\Rightarrow ar^2 + br + c = 0.$

We wish to find solutions $(x_1, \ldots, x_n)$ to the system

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\
&\vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.
\end{aligned}
$$

We know from Ch 5.1 that we can write this more compactly as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \text{or} \qquad \frac{d\vec{x}}{dt} = \vec{A}\vec{x}$$

- Make the following guess: $\mathbf{x} = \mathbf{v}e^{\lambda t}$ $= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix} e^{\lambda t} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ v_3 e^{\lambda t} \\ \vdots \end{pmatrix}$
- Substitute into DE, giving $\lambda \vec{v} e^{\lambda t} = \vec{A}\vec{v}e^{\lambda t}$
- Therefore $\mathbf{v}$ and λ solve $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ So $\lambda\vec{v} = \vec{A}\vec{v}$

We've simplified the problem to an algebraic problem.

How to solve this algebraic problem? $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

unknowns what we want to solve for.

This is called an eigenvalue/eigenvector problem.

The λ solutions are called *eigenvalues* of $\mathbf{A}$

The $\mathbf{v}$ solutions are called *eigenvectors* of $\mathbf{A}$

- Use characteristic polynomial

Example 1 Find a general solution of the system

$$x_1' = 4x_1 + 2x_2,$$
$$x_2' = 3x_1 - x_2.$$

Solution The matrix form of the system in (11) is

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x}. \quad (12)$$

1. Guess that solution is of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Substitute into (12).

$$\lambda \vec{v} \cancel{e^{\lambda t}} = \vec{A} \vec{v} \cancel{e^{\lambda t}}$$

2. Get an eigenvalue problem.

$$\lambda \vec{v} = \vec{A} \vec{v}$$

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

3. Solve the eigenvalue problem.

$$\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}.$$

i.e. $4v_1 + 2v_2 = \lambda v_1$
$3v_1 - v_2 = \lambda v_2$

By linear algebra, $\lambda = -2, 5$. (see next slides).

if $\lambda = -2$: $\begin{cases} 4v_1 + 2v_2 = -2v_1 \\ 3v_1 - v_2 = -2v_2 \end{cases}$

$\Rightarrow \begin{cases} 6v_1 + 2v_2 = 0 \\ 3v_1 + v_2 = 0. \end{cases}$

$\Rightarrow v_2 = -3v_1$. One solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

if $\lambda = 5$.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Check: $\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Example 1 Find a general solution of the system

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Solution The matrix form of the system in (11) is

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x}. \quad (12)$$

1. Guess that solution is of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Substitute into (12).

2. Get an eigenvalue problem.

3. Solve the eigenvalue problem.

$\lambda = -2$, $v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

and

$\lambda = 5$, $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

equivalent $c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^t$

$c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^t$

not equivalent to

$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^t$

4. Use eigenvectors to write down solution to the DE.

$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ $\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$

So general soln is

$c_1\vec{x}_1 + c_2\vec{x}_2$.

E.g: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The $n \times n$ identity matrix is the matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$I_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

It is useful because for all matrices $\mathbf{A}$, we have $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

To solve the eigenvalue problem $\mathbf{Av} = \lambda\mathbf{v}$, we can first find the eigenvalues by solving the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

characteristic polynomial of A.

for the previous example:

1. Make the matrix $\mathbf{A} - \lambda\mathbf{I}$

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix}.$$

2. Find determinant of $\mathbf{A} - \lambda\mathbf{I}$

$$\det(A - \lambda I) = (4-\lambda)(-1-\lambda) - 6$$
$$= -4 + \lambda^2 + \lambda - 4\lambda - 6$$
$$= \lambda^2 - 3\lambda - 10.$$

3. Solve for roots λ.

$$\lambda = -2, 5.$$

Works exactly the same for larger systems:

$$\begin{aligned} x_1' &= -k_1 x_1, \\ x_2' &= \quad k_1 x_1 - k_2 x_2, \\ x_3' &= \qquad\qquad k_2 x_2 - k_3 x_3, \end{aligned}$$

$k_1 = 0.5$
$k_2 = 0.25$
$k_3 = 0.2$

1. Rewrite in matrix form $\vec{x}' = \vec{A}\vec{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{Av} = \lambda\mathbf{v}$
3. Find the eigenvalues of A
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, use initial conditions if applicable.

①

$$\mathbf{x}'(t) = \underbrace{\begin{bmatrix} -0.5 & 0.0 & 0.0 \\ 0.5 & -0.25 & 0.0 \\ 0.0 & 0.25 & -0.2 \end{bmatrix}}_{A} \mathbf{x},$$

② $\vec{A}\vec{v} = \lambda\vec{v}$

③ 1. $\det\underbrace{\begin{pmatrix} -0.5-\lambda & 0 & 0 \\ 0.5 & -0.25-\lambda & 0 \\ 0 & 0.25 & -0.2-\lambda \end{pmatrix}}_{A-\lambda I}$

$= (-0.5-\lambda)\det\begin{pmatrix} -0.25-\lambda & 0 \\ 0.25 & -0.2-\lambda \end{pmatrix}$

$- 0 + 0$

$= (-0.5-\lambda)(-0.25-\lambda)(-0.2-\lambda)$

2. roots are $\lambda = -0.5$
$\lambda = -0.25$
$\lambda = -0.2.$

Works exactly the same for larger systems:

$$\begin{aligned} x_1' &= -k_1x_1, \\ x_2' &= \quad k_1x_1 - k_2x_2, \\ x_3' &= \qquad\qquad k_2x_2 - k_3x_3, \end{aligned}$$

1. Rewrite in matrix form
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, use initial conditions if applicable.

Eigenvalues are $\lambda_1 = -0.5$, $\lambda_2 = -0.25$, $\lambda_3 = -0.2$.

④ if $\lambda = -0.5$:

$$A\vec{v} = \lambda\vec{v}$$

$$\Downarrow$$

$$\begin{pmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = -0.5\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -0.5v_1 = -0.5v_1 & \leftarrow \text{useless.} \\ 0.5v_1 - 0.25v_2 = -0.5v_2 & (2) \\ 0.25v_2 - 0.2v_3 = -0.5v_3 & (3) \end{cases}$$

Underdetermined, so
Take $v_1 = 1$
plug into (2), (3), solve for v_2, v_3,
get $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 5/3 \end{bmatrix}$.

After more algebra:

$\lambda_1 = -0.5$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 5/3 \end{bmatrix}$.

$\lambda_2 = -0.25$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

$\lambda_3 = -0.2$. $\quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Solutions to DE:

$\vec{x}_1(t) = \begin{bmatrix} 1 \\ -2 \\ 5/3 \end{bmatrix} e^{-0.5t}$

$\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} e^{-0.25t}$

$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-0.2t}$.

Today:

- How to reduce the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  to the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.
- How to solve the eigenvalue problem for some eigenvalues and eigenvectors.

Actually, sometimes we won't get n real eigenvectors.
There could be missing solutions, or some them could be complex.

We'll talk about how to deal with those cases next time. (Ch 5.2, 5.5).