# MAT303: Calc IV with applications

Lecture 2 - February 08 2021

- What is a differential equation
- Why we should study differential equations
- Ch1.1: Differential equations and mathematical models
- Ch1.2: Integrals as solutions to differential equations

$y = \int f(x)\,dx + C$

$\frac{dy}{dt} = f(t)$ $\qquad$ $\frac{dy}{dx} = f(x)$.

$y(0)$ is specified.

What/why:

- Many processes in the world can be described by their rate of change
- Rate of change <-> derivative
- Equations involving derivatives are *differential equations*.
- Differential equations allow us to study mathematical models of physical processes.

$$\begin{cases} \frac{dy}{dx} = y \\ \text{Soln: } y = Ce^x \\ y(0) = 2 \Rightarrow C = 2. \end{cases}$$

Today:

- Different ways of interpreting functions and DEs
- Slope field

Advantages of multiple interpretations

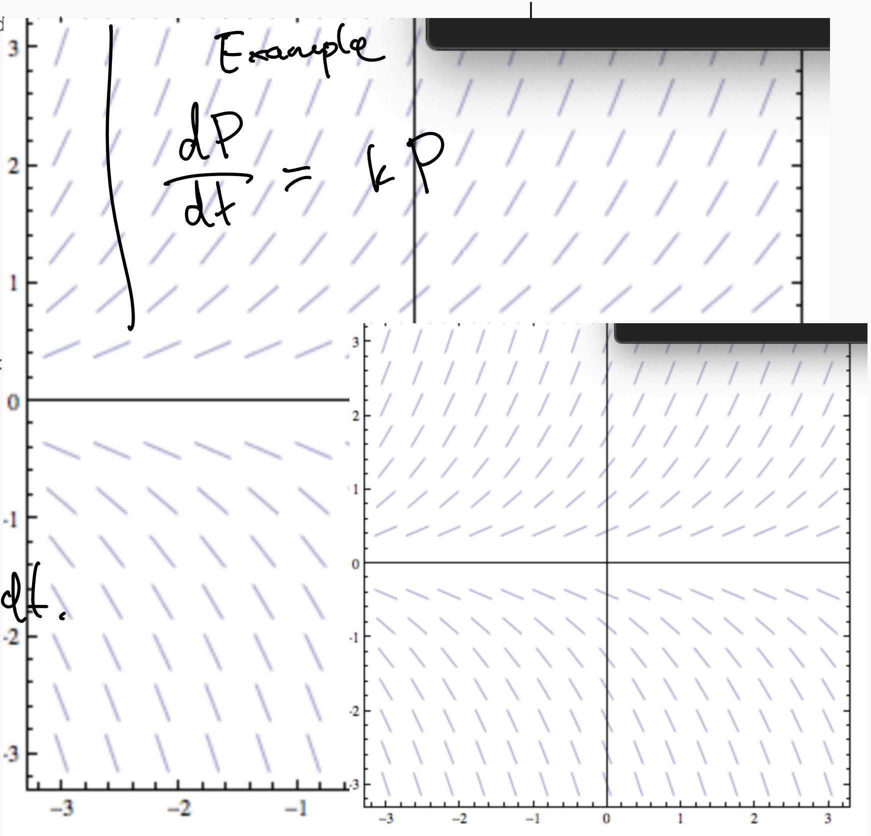
- More opportunities to see when DEs are useful
- Easy to reason about general properties of DEs
- Easy to reason about specific DEs

We will see:

- Why most DE has infinitely many solutions
- Why adding an initial condition makes it unique

First order equation where RHS does not d

$$\frac{dy}{dt} = f(y,t)$$




Example

$$\frac{dP}{dt} = kP$$

First order equation with initial condition:

$$\frac{dy}{dt} = f(t)$$

$$y = \int f(t)\, dt.$$

First order differential equation:

$$\frac{dy}{dx} = f(x, y) \quad \left(\text{or } \frac{dy}{dt} = f(t, y)\right).$$

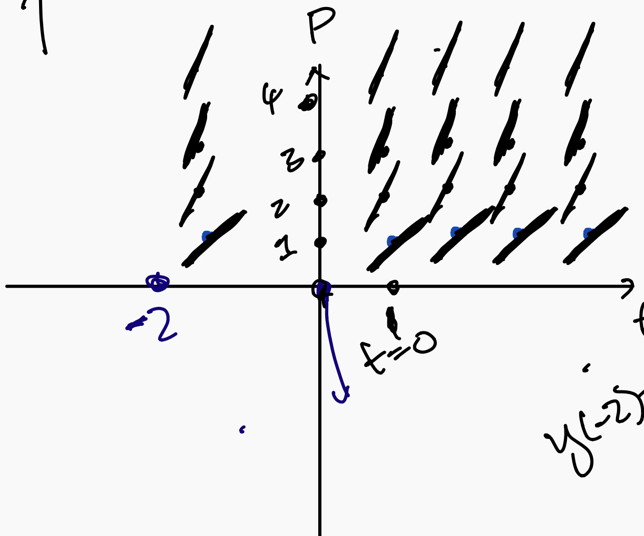
→ Find a function whose slope is given by that formula.

Draw the slope field:

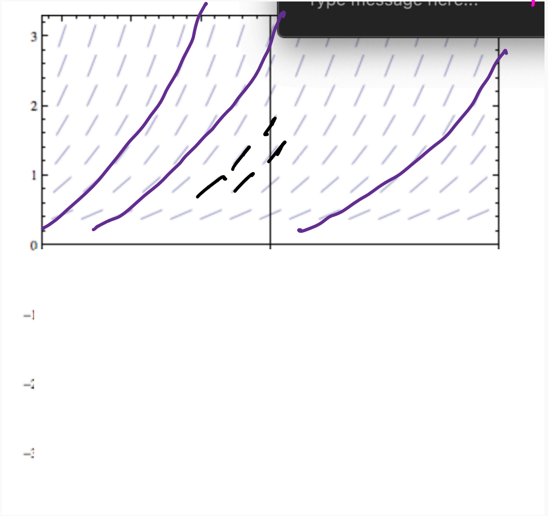
$$\frac{dP}{dt} = P$$

$P = e^t$ is soln

$$\frac{dP}{dt} = e^t = P$$




$y(-2) = C$



$$\frac{dy}{dx} = x - y$$

last:
$$\frac{dP}{dt} = P.$$

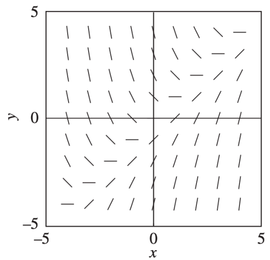


FIGURE 1.3.4. Slope field for $y' = x - y$ corresponding to the table of slopes in Fig. 1.3.3.

$x=1$
$y=2.$
$\frac{dy}{dx}=-1$

Initial condition:

y(-4)=4

$y=4$ when $x=-4.$

$$\frac{dy}{dx} = x-y$$

$y=x^2$

$2x = x-x^2$

not true.

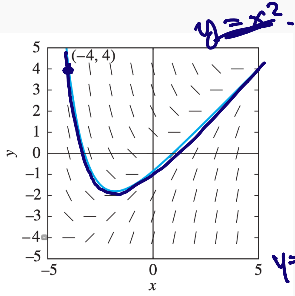


FIGURE 1.3.5. The solution curve through $(-4, 4)$.

$y=x^2.$

$(x=1$
$y=1)$

slope of $y=x^2$ is 2.

$\frac{dv}{dt} = a$

a) k negative
b) k positive.
c) I don't know.

Air resistance is proportional to velocity: $v > 0$.

$$\frac{dv}{dt} = g - kv$$

32ft/s$^2$

$$\frac{dv}{dt} = 32 - 0.16v$$

Ball being dropped from a plane.

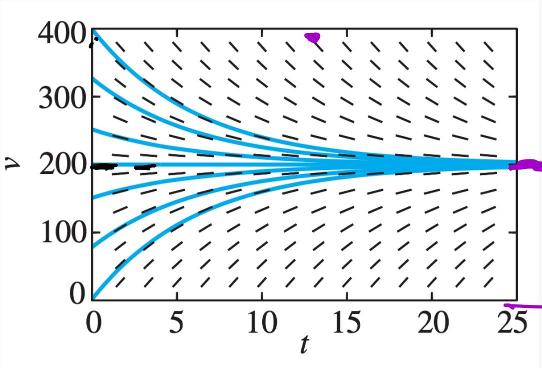
9.8 m/s$^2$

Can someone catch it?

Or how fast it is on impact

Conclusion

Velocity will be ~200ft/s.




E.g. $v = 200$, $\frac{dv}{dt} = 0$

$v > 200$, $\frac{dv}{dt} < 0$

$v < 200$, $\frac{dv}{dt} > 0$.

parameters

old Population growth model.

$$\frac{dP}{dt} = kP$$

Logistic model for population growth:

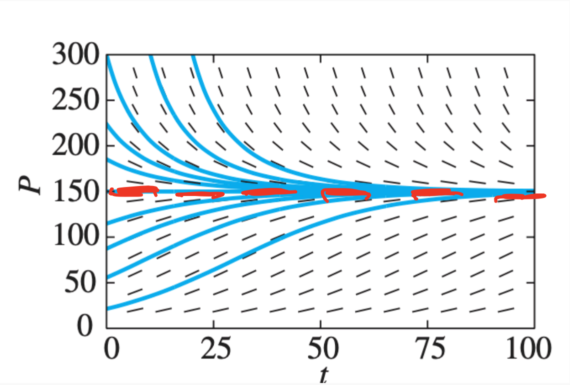
$$\frac{dP}{dt} = kP(M - P)$$

$$\frac{dP}{dt} = 0.0004P(150 - P)$$

k M

M is total pop.
P is infected population.

Conclusion:
Pop tends to 150.




When $P > 150$, $\frac{dP}{dt} < 0$

$P = 150$, $\frac{dP}{dt} = 0$.

$P < 150$ $\frac{dP}{dt} > 0$.

$$\frac{dP}{dt} = P \qquad P = Ce^{t}$$

Intuitively:

- Differential equations usually have infinitely many solutions
- Adding an initial condition usually narrows it down to a unique solution

$$\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + C$$

$$y(0) = 1 \Rightarrow C = 1.$$

The technical statement:

**THEOREM 1 Existence and Uniqueness of Solutions**

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are continuous on some rectangle $R$ in the $xy$-plane that contains the point $(a, b)$ in its interior. Then, for some open interval $I$ containing the point $a$, the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(a) = b \tag{9}$$

has one and only one solution that is defined on the interval $I$. (As illustrated in Fig. 1.3.11, the solution interval $I$ may not be as "wide" as the original rectangle $R$ of continuity; see Remark 3 below.)

Example:

$$\frac{dy}{dx} = f(x, y)$$

$$x\frac{dy}{dx} = 2y$$