MAT303: Calc IV with applications

Lecture 18 - April 12 2021

So far this class:

- Looking at single differential equations

Rest of the class:

- Systems of differential equations (analogous to systems of algebraic equations)

Consider the system

$$x' = 4x - 3y \qquad (1)$$
$$y' = 6x - 7y \qquad (2)$$

Analogously to algebraic systems, we can try eliminating one of the variables:

Rearrange 2nd eqn:

$$6x = 7y + y' \Rightarrow x = \frac{7}{6}y + \frac{y'}{6} \qquad (3)$$

$$x' = \frac{7}{6}y' + \frac{y''}{6}$$

Sub into (1):

$$\frac{7}{6}y' + \frac{y''}{6} = 4\left(\frac{7}{6}y + \frac{y'}{6}\right) - 3y$$

$$7y' + y'' = 28y + 4y' - 18y$$

$$\Rightarrow y'' + 3y' - 10y = 0.$$

So $r^2 + 3r - 10$

So $r_{1,2} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2}$

So $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

Plug this into (3) to get

$$x = \frac{7}{6}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) + \frac{1}{6}\left(r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}\right).$$

Consider the constant coefficient linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y \cancel{= 0} \qquad (1)$$

We can rewrite this as

$$Ly \cancel{= 0}$$

where $L = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_0$ is an **operator**

where $D = \dfrac{d}{dx}$ is the **derivative operator**

Examples of operator notation:

E.g. $Df = f'$

$D^2 f = f''$

$a_0 Df = a_0 f'$

$(D+3)f = f' + 3f.$

$f(x) = x \to 1 + 3x$

"Multiplication" of differential operators

$L_1 L_2 f$ means $L_1(L_2 f) \rightarrow$ do L_2 first, then L_1.

Constant coefficient polynomial differential operators are commutative:

$$L_2 L_1 = L_1 L_2$$

E.g.

$L_1 = D + 3$

$L_2 = D + 4$

$$L_1 L_2 f = (D+3)(D+4) f$$
$$= (D^2 + 7D + 12) f$$
$$= f'' + 7f' + 12 f$$

Consider the system

$$x' = 4x - 3y$$
$$y' = 6x - 7y$$

We will solve this more systemically, using differential operators.

"D" means "Take derivative of".

Rewrite this as:

$$-4x + x' + 3y = 0$$
$$-6x + 7y + y' = 0$$

$$-4x + Dx + 3y = 0$$
$$-6x + 7y + Dy = 0.$$

Rewrite this as:

$$(-4 + D)x + 3y = 0$$
$$-6x + (7 + D)y = 0$$

Rewrite this as:

$$L_1x + L_2y = 0$$
$$L_3x + L_4y = 0$$

Where $L_1 = D - 4,$ $L_2 = 3,$ $L_3 = -6,$ $L_4 = D + 7$

Operate on top equation by L_3 and bottom equation by L_1 :

$$L_3L_1x + L_3L_2y = 0$$
$$L_1L_3x + L_1L_4y = 0$$

Subtract top equation from bottom:

$$L_1L_4y - L_3L_2y = 0$$

This is a single variable equation which we can solve.

$$(D-4)(D+7)y - (-6)(3)y = 0$$

$$(D^2 + 3D - 28)y + 18y = 0.$$

So $D^2y + 3Dy - 28y + 18y = 0$

So $y'' + 3y' - 10y = 0.$

Same as what we had before.

In the previous example, the key milestone was rewriting the equation in the form

$$L_1x + L_2y = 0$$
$$L_3x + L_4y = 0$$

$L y_1 + L y_2 = 0$

$L_3 y_1 + L y_2 = 0$

In general, whenever we can write our system in the form

$$L_1x + L_2y = f_1(t)$$
$$L_3x + L_4y = f_2(t)$$

we can do the same trick:

- Operate on top equation by L_3 and bottom equation by L_1 :

$$L_3L_1x + L_3L_2y = L_3f_1(t)$$
$$L_1L_3x + L_1L_4y = L_1f_2(t)$$

Subtract top equation from bottom:

$$\boxed{L_1L_4y - L_3L_2y = L_1f_2(t) - L_3f_1(t)}$$

This is a single variable equation which we can solve for y.

(only 1 unknown function).

- Operate on top equation by L_4 and bottom equation by L_2 :

$$L_4L_1x + L_4L_2y = L_4f_1(t)$$
$$L_2L_3x + L_2L_4y = L_2f_2(t)$$

Subtract bottom equation from top:

$$\boxed{L_4L_1x - L_2L_3x = L_4f_2(t) - L_2f_1(t)}$$

This is a single variable equation which we can solve for x.

Summary:

The solution to

$$L_1x + L_2y = f_1(t)$$
$$L_3x + L_4y = f_2(t)$$

is given by solving

$$\boxed{(L_4L_1 - L_2L_3)x = L_4f_2(t) - L_2f_1(t)}$$

and

$$\boxed{(L_1L_4 - L_3L_2)y = L_1f_2(t) - L_3f_1(t)}$$

Example: suppose want to solve the system

$$\begin{aligned} x' - 4x + 3y &= 0 \\ -6x + y' + 7y &= 0 \end{aligned}$$

Rewrite in terms of of differential operators:

$$\begin{aligned} Dx - 4x + 3y &= 0 \\ -6x + Dy + 7y &= 0 \end{aligned} \quad \equiv \quad \begin{aligned} \overbrace{(D-4)}^{L_1}x + \overbrace{3}^{L_2}y &= 0 \\ \underbrace{-6}_{L_3}x + \underbrace{(D+7)}_{L_4}y &= 0 \end{aligned}$$

According to what we just did:
(on the right)

$$* \; L_4L_1 - L_2L_3 = (D+7)(D-4) - (-6)(3)$$
$$= D^2 + 3D - 10$$

So the 2 eqns we need to solve are

$$\begin{cases} (D^2+3D-10)x = 0 \\ (D^2+3D-10)y = 0. \end{cases}$$

$$\Rightarrow \begin{cases} x'' + 3x' - 10x = 0 \\ y'' + 3y' - 10y = 0. \end{cases}$$

Summary:

The solution to

$$\begin{aligned} L_1x + L_2y &= f_1(t) \\ L_3x + L_4y &= f_2(t) \end{aligned}$$

is given by solving

$$\underline{(L_4L_1 - L_2L_3)}x = L_4f_2(t) - L_2f_1(t)$$

and

$$\underline{(L_1L_4 - L_3L_2)}y = L_1f_2(t) - L_3f_1(t)$$

To solve 2x2 linear equation:

$$
\begin{aligned}
ax + by &= p \\
cx + dy &= q
\end{aligned}
$$

We can use Cramer's rule:

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$x = \frac{pd - qb}{ad - bc}, \quad y = \frac{aq - cp}{ad - bc}$$

very similar.

Summary (to solve 2x2 first order system):

The solution to

$$
\begin{aligned}
L_1 x + L_2 y &= f_1(t) \\
L_3 x + L_4 y &= f_2(t)
\end{aligned}
$$

is given by solving

$$(L_4 L_1 - L_2 L_3)x = L_4 f_2(t) - L_2 f_1(t)$$

and

$$(L_1 L_4 - L_3 L_2)y = L_1 f_2(t) - L_3 f_1(t)$$

Summary

Today:

- Solving systems of first order equations by elimination
- Using polynomial differential operators

Next time: Chapter 5.1

- Better methods for larger systems