MAT303: Calc IV with applications

Lecture 17 - April 5 2021

So far this class:

- Looking at single differential equations

Rest of the class:

- Systems of differential equations (analogous to systems of algebraic equations)

Individual algebraic equation

$$3x = 1$$

$$\Rightarrow \quad x = \frac{1}{3}$$

Systems of algebraic equations

$$\begin{cases} 3x + 2y = 5 \\ 2x + y = 3 \end{cases}$$

$$\Rightarrow (x, y) = (1, 1)$$

Individual differential equation

$$x'(t) = x(t)$$

$$\Rightarrow \quad x(t) = e^t$$

Systems of differential equations

$$\begin{cases} x'(t) = -2y(t) \\ y'(t) = \frac{1}{2} x(t) \end{cases}$$

$$\Rightarrow (x(t), y(t)) = (\cos t, \tfrac{1}{2} \sin t)$$

Check:

$$x'(t) = -\sin t = -2y(t)$$

$$y'(t) = \frac{1}{2} \cos t = \frac{1}{2} x(t).$$

Spring-mass problems

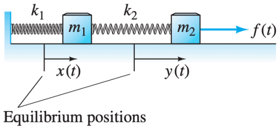


FIGURE 4.1.1. The mass-and-spring system of Example 1.

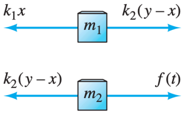


FIGURE 4.1.2. The "free body diagrams" for the system of Example 1.

system of DEs.

Mixing problems

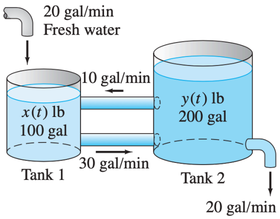


FIGURE 4.1.3. The two brine tanks of Example 2.

$x(t)$ = salt in tank 1

$y(t)$ = salt in tank 2.

$$x'(t) = \text{salt in} - \text{salt out} \quad (\text{tank } 1)$$

$$= (\text{conc. in}) \times (\text{water in}) - (\text{conc. out}) \times (\text{water out})$$

$$= \frac{y(t)}{200+30t-10t-20t} 10 - \frac{x(t)}{100-30t+10t+20t} 30$$

$$= \frac{y(t)}{20} - \frac{3x(t)}{10}$$

$$y'(t) = \frac{3x(t)}{10} - \frac{3}{20} y(t)$$

Aside from the previous applications, systems of differential equations naturally arise when we consider higher order DEs.

Example: The 3rd order equation

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

is equivalent to the system of 3 equations

$$\left.\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{aligned}\right\}$$

First order system

How?

Introduce new variables. Let

$$\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = x''. \end{cases}$$

$x_1 = x$

Diff both sides:

$x_1' = x'$

so $x_1' = x_2$

$x_2 = x'$

so $x_2' = x'' = x_3$

Then

$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$

becomes

$$\begin{cases} x_3' + 3x_3 + 2x_2 - 5x_1 = \sin 2t \\ x_2' = x_3 \\ x_1' = x_2 \end{cases}$$

In general: Any higher order differential equation can be transformed into a system of <u>first order</u> equations, by introducing new variables.

Example: The system of 2nd order DEs

$$\begin{aligned} 2x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40 \sin 3t \end{aligned}$$

is equivalent to the system of 4 first order equations:

$$\begin{aligned} x_1' &= x_2 \\ 2x_2' &= -6x_1 + 2y_1 \\ y_1' &= y_2 \\ y_2' &= 2x_1 - 2y_1 + 40 \sin 3t \end{aligned}$$

Let $x_1 = x$ $\quad y_1 = y$

$x_2 = x'$ $\quad y_2 = y'$

Then

$$\begin{cases} 2x_2' = -6x_1 + 2y_1 \\ y_2' = 2x_1 - 2y_1 + 40 \sin 3t \end{cases}$$

We've now seen how systems of first order equations naturally arise:

- Directly from applications
- From (systems of) higher order equations

In the rest of this semester we will look at various methods to solve them.

- Many methods will involve linear algebra/matrices.

Consider the system

$$y(t) = -\tfrac{1}{2}x'(t) = \tfrac{1}{2}C\sin(t-\alpha)$$

(1) $x'(t) = -2y(t)$

(2) $y'(t) = \frac{1}{2}x(t)$

$$y'(t) = \tfrac{1}{2}C\cos(t-\alpha)$$
$$\Rightarrow y(t) = \tfrac{1}{2}C\sin(t-\alpha) + \cancel{D}$$

We will solve this without any linear algebra. It relies on what we already know for second order DEs.

Analogously to algebraic systems, we can try eliminating one of the variables:

$x' = -2y$. sub in (2)

diff. both sides

$$\Rightarrow x'' = -2y' = -2\left(\tfrac{1}{2}x\right)$$

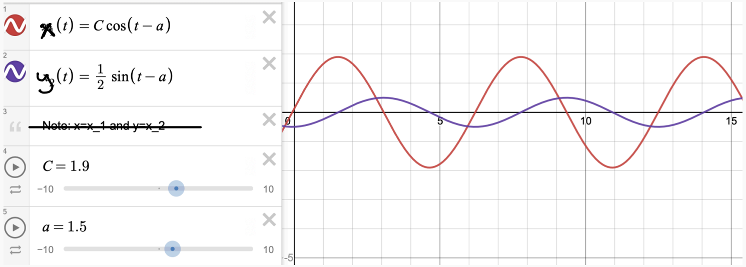
We've eliminated y.

So $x'' = -x$.

sub back in

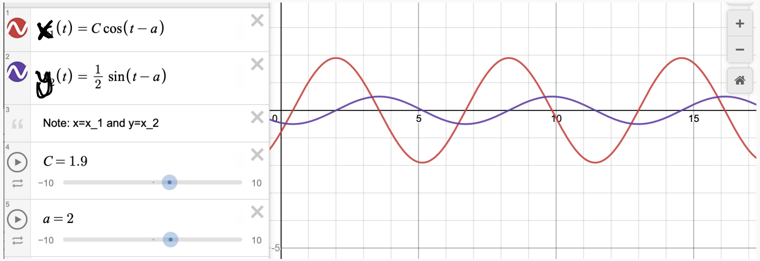
Soln is $x(t) = A\cos t + B\sin t$
$= C\cos(t-\alpha)$.

~~Solution:~~ Solutions: $\left(x(t) = C\cos(t-\alpha), \quad y(t) = \frac{1}{2}C\sin(t-\alpha)\right)$



Changing C changes the amplitude.

Changing α translates the functions:



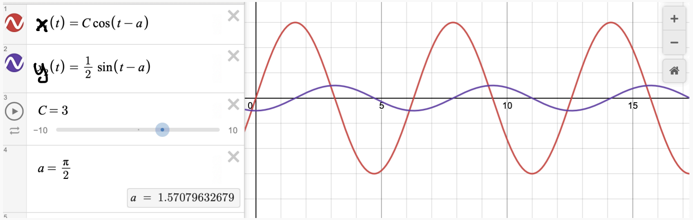
Consider the system

$$x'(t) = -2y(t)$$
$$y'(t) = \frac{1}{2}x(t)$$

$x(0) = 3\cos(-\frac{\pi}{2}) = 0$

$y(0) = \frac{1}{2} 3 \sin(-\frac{\pi}{2}) = -3/2$

Solution: $x(t) = C\cos(t-\alpha), \quad y(t) = \frac{1}{2}C\sin(t-\alpha)$



$\cos^2(t-\alpha)$ $\quad$ $\sin^2(t-\alpha)$

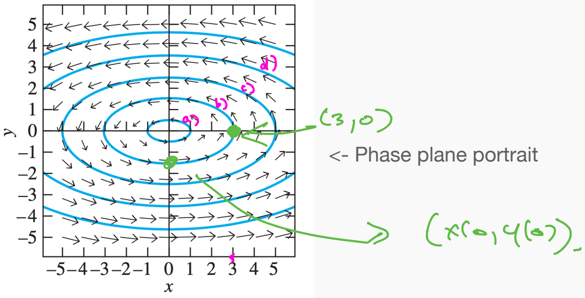
$$\left(\frac{x}{C}\right)^2 + \left(\frac{2y}{C}\right)^2 = 1$$

$C=3$, then $(x=3, y=0)$ is on trajectory.

There is another way to think about the solution.

The state of the system is described by a path $(x(t), y(t))$ through the plane $\mathbb{R}^2$.

For this DE, the solution curves trace out ellipses in the state space.



<- Phase plane portrait

FIGURE 4.1.6. Direction field and solution curves for the system $x' = -2y$, $y' = \frac{1}{2}x$ of Example 6.

Make sure you understand how this different the direction fields we considered in Ch1. There is no time axis.

Test your understanding:

- Which curve corresponds to the solution on the left? b)
- What is the effect does changing C have?
- What is the effect does changing α have?
- How did we draw the direction field?
- How can we determine that the trajectories are actually ellipses?

Consider the system

$$x' = y$$
$$y' = 2x + y$$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = 2$

chain rule $\frac{dy}{dx} = 1$

Analogously to algebraic systems, we can try eliminating one of the variables:

$x' = y$

$\Rightarrow x'' = y' = 2x + y = 2x + x'$

$\Rightarrow x'' = 2x + x'$

$\Rightarrow x'' - x' - 2x = 0$

$\Rightarrow r^2 - r - 2 = 0$

$\Rightarrow r = -1, r = 2.$

$\Rightarrow x(t) = Ae^{-t} + Be^{2t}$

$\Rightarrow y(t) = x'(t) = -Ae^{-t} + 2Be^{2t}.$

Solution: $x(t) = Ae^{-t} + Be^{2t}$, $y(t) = -Ae^{-t} + 2Be^{2t}$

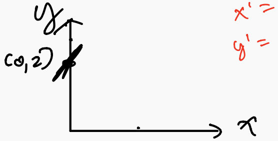
$x(t) = e^{2t}, \quad y(t) = 2e^{2t}.$

Quiz: at $(x, y) = (0,2)$ what is the slope of the direction field?

a) 1
b) 2
c) 3
d) Don't know

at (0,2) $x' = 2$

at (0,2) $y' = 2$



direction is (2,2).

Length also matters.

Phase portrait:

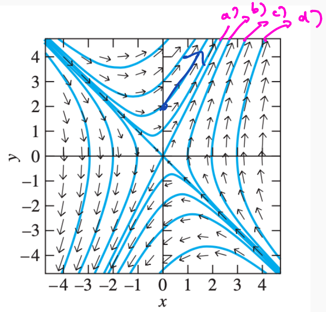


FIGURE 4.1.8. Direction field and solution curves for the system $x' = y$, $y' = 2x + y$ of Example 7.

Quiz: which curve corresponds to $A = 0, B = 1$?

Convert to cartesian: $y = 2x$. Answer is a). Or plug in $t=0, t=1, t=2$

~~Quiz: which curve corresponds to $A = 0, B = 1$?~~

Quiz: what direction is the trajectory taking?

Consider the system

$$x' = -y$$
$$y' = 1.01x - 0.2y$$

Analogously to algebraic systems, we can try eliminating one of the variables:

$$x' = -y$$

$$\Rightarrow x'' = -y' = -1.01x + 0.2y = -1.01x - 0.2x'$$

$$\Rightarrow \quad x'' + 0.2x' + 1.01x$$

$$\Rightarrow \quad r^2 + 0.2r + 1.01$$

$$\Rightarrow r = -0.1 \pm \sqrt{0.01 - 1.01}$$
$$= -0.1 \pm i$$

$$\Rightarrow x(t) = Ae^{-t/10}\sin t + Be^{-t/10}\cos t$$

$$\Rightarrow y(t) = \frac{1}{10}Ae^{-t/10}\cos t - \frac{1}{10}Be^{-t/10}\sin t$$

Solution: $x(t) = e^{-t/10}\sin t, \quad y(t) = \frac{1}{10}e^{-t/10}(\sin t + 10\cos t)$

Individual solutions:

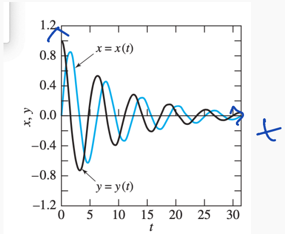


FIGURE 4.1.10. x- and y-solution curves for the initial value problem of Example 8.

<- Only shows one solution

Phase portrait:

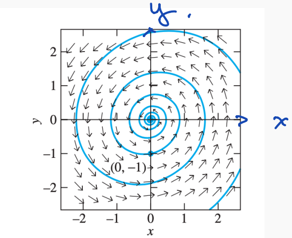


FIGURE 4.1.9. Direction field and solution curve for the system $x' = -y$, $y' = (1.01)x - (0.2)y$ of Example 8.

<- Shows all solutions,
But doesn't contain information
about the time parameterization

- Systems of first order equations are important because
  - They arise naturally in applications
  - Any higher order equation can be transformed to such a system

$n=2$ today.

- The solution to a system with n equations can be viewed as
  - n different functions, or
  - A single parametric curve through $\mathbb{R}^n$
- It is useful to draw direction fields on the phase portraits for a system of DEs.
  - If there are n unknown functions, then the phase space is $\mathbb{R}^n$

Example (logistic equation, $n = 1$): $\dfrac{dP}{dt} = P(100 - P)$

Example (predator-prey): $$x'(t) = -2y(t)$$ $$y'(t) = \frac{1}{2}x(t)$$

$n=2$.

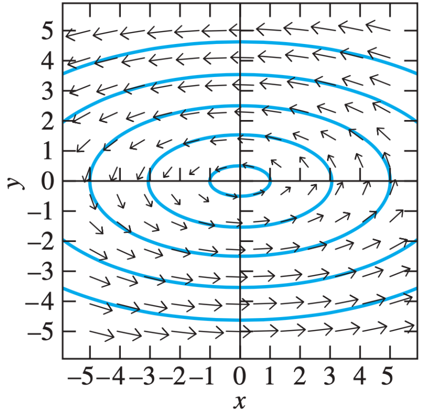


FIGURE 4.1.6. Direction field and solution curves for the system $x' = -2y$, $y' = \frac{1}{2}x$ of Example 6.