MAT303: Calc IV with applications

Lecture 16 - March 31 2021

Recently: Initial Value Problems

- Looking at $my'' + cy' + ky = f(t)$

  where we are solving for $y(t)$, $y(0) = a, y'(0) = b$

Today: Endpoint problems (Ch 3.8)

- $y'' + p(x)y' + q(x)y = 0$, where we are solving for $y(x)$, $y(0) = a$, $y(1) = b$
- Key difference: boundaries conditions of the form ~~$y(a) = \cdots, y(b)$~~ $= \cdots$

$y(c) = \ldots$, $y(d) = \ldots$

$c \neq d$.

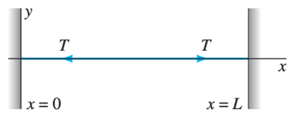
The shape of a jump rope being twirled satisfies the following equation:

$$
\begin{aligned}
Ty'' + \rho\omega^2 y &= 0 \\
\left.\begin{aligned} y(0) &= 0 \\ y(L) &= 0 \end{aligned}\right\} &\text{ endpoints fixed.}
\end{aligned}
$$

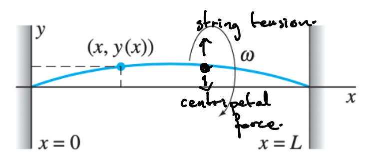
Where

- ω is angular frequency (constant)
- ρ is density fo the string (mass per unit length, constant)
- L is the length of the string (constant)
- y is the radial deviation from the center
- $0 \leq x \leq L$ is the location along the string we are measuring the deviation
- T is the tension force due to spring (constant) tangential to string.

(b)



Equilibrium position



Whirling string

(b)

End PoV.

Explanation of the equation (not a full derivation): slope.

- Net y component of tension force is Ty''
  - y component of force is proportional to y'
  - So net y component of force is prop. to y''

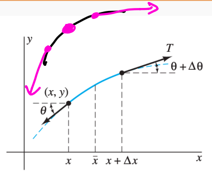


FIGURE 3.8.7. Forces on a short segment of the whirling string.

angular frequency.

dist to center

- Centripetal force is $-\rho\omega^2 y$
  - Further away from center -> bigger circle -> greater force
  - Faster spin -> greater force.

When stable, forces should be balanced, $Ty'' = -\rho\omega^2 y$.

We see that it is natural to have boundary conditions at different points 0 and L, instead of at the same point.

Instead of $Ty'' + \rho\omega^2 y = 0$
$y(0) = 0$
$y(L) = 0$

Look at $\underbrace{y'' + 3y = 0}$; $y(0) = 0$, $y(\pi) = 0$ (For simplicity)

Solution: characteristic eqn is $r^2+3=0$
$\Rightarrow r = \pm\sqrt{3}\, i$

So general soln is
$y = A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x)$

$y(0)=0 \Rightarrow A\cdot 1 + B\cdot 0 = A \Rightarrow A=0$
$\Rightarrow y(x) = B\sin(\sqrt{3}x)$.

$y(\pi)=0 \Rightarrow B\sin(\sqrt{3}\pi)=0 \Rightarrow B=0$.

Soln is $y=0$. Only trivial solns.

Look at $y'' + 4y = 0$; $y(0) = 0$, $y(\pi) = 0$ (For simplicity)
characteristic eqn is $r^2+4=0$
$\Rightarrow r = \pm 2i$

Solution: So general soln is
$y = A\cos(2x) + B\sin(2x)$

$y(0)=0 \Rightarrow A\cdot 1 + B\cdot 0 = A \Rightarrow A=0$
$\Rightarrow y(x) = B\sin(2x)$.

$y(\pi)=0 \Rightarrow B\underbrace{\sin(2\pi)}_{0}=0 \Rightarrow B$ can be anything.

So solutions are $y = B\sin(2x)$, any $B\in\mathbb{R}$.

We see that $y'' + \lambda y = 0$; $y(0) = 0$, $y(\pi) = 0$
only has non zero solutions for certain λ.

Let's find out when there are solutions (i.e. let's solve the "eigenvalue problem"):

Leave λ as a variable:
characteristic eqn is $r^2+\lambda=0$
$\Rightarrow r = \pm\sqrt{\lambda}\, i$

general soln is
$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$(0)=0 \Rightarrow A\cdot 1 + B\cdot 0 = A \Rightarrow A=0$
$\Rightarrow y(x) = B\sin(\sqrt{\lambda}x)$.

$y(\pi)=0 \Rightarrow B\sin(\sqrt{\lambda}\pi)=0$
* if $\sin(\sqrt{\lambda}\pi)\neq 0$, $B=0$
* if $\sin(\sqrt{\lambda}\pi)=0$, B can be anything.
i.e. $\sqrt{\lambda}=n$ integer, i.e. $\lambda=n^2$.

Infinitely many solutions $y = B\sin(\sqrt{\lambda}\pi)$

Contrast to initial value problem, where every choice of λ has a solution:

$$y'' + \lambda y = 0;\quad y(0) = 0,\quad y'(0) = 0$$

unique.

Recall from linear algebra:

Let A be a matrix (linear operator)

If we wish to find vectors v and numbers λ such that

$$Av = \lambda v$$

This is called an eigenvalue problem.

Solutions might only exists for certain λ and certain v.

Alternatively, you're solving for both λ and v.

If (λ, v) is a solution, λ is called an eigenvalue and v is called an eigenvector.

Note: if v is an eigenvector, so is any constant multiple of v.

if $Av = \lambda v$, let $w = kv$,

$Aw = Akv = kAv = k\lambda v = \lambda kv = \lambda w.$

$$\underbrace{y'' + \lambda y = 0}; \quad y(0) = 0, \quad y(\pi) = 0$$

i.e. $y'' = -\lambda y$

i.e. $Ly = -\lambda y$.

Eigenvalues: $\lambda_n = n^2$

$\lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 9 \ldots$

$\lambda = (\frac{1}{2}+n)^2$

Solve the eigenvalue problem (For simplicity)

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y'(\pi) = 0$$

Solution: Find eigenvalues λ and eigenfunctions y.

Leave λ as a variable:

characteristic equ is $r^2 + \lambda = 0$

$\Rightarrow r = \pm\sqrt{\lambda}\, i$

general sol'n is

$y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

$y(0)=0 \Rightarrow A\cdot 1 + B\cdot 0 = A \Rightarrow A=0$

$\Rightarrow y(x) = B\sin(\sqrt{\lambda}x)$.

Now $y'(x) = \sqrt{\lambda}B\cos(\sqrt{\lambda}x)$

so $y'(\pi) = \sqrt{\lambda}B\cos(\sqrt{\lambda}\pi) = 0$

If $\cos(\sqrt{\lambda}\pi) = 0$, B can be anything.

$\Rightarrow \sqrt{\lambda}\pi = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

$\Rightarrow \sqrt{\lambda} = \frac{1}{2} + n$, $n \in \mathbb{Z}$

$\Rightarrow \lambda_n = (\frac{1}{2}+n)^2$, $n \in \mathbb{Z}$

if $\lambda \neq (\frac{1}{2}+n)^2$, then $B = 0$,

so $y = 0$.

Eigenvalues are: $\lambda_n = (\frac{1}{2}+n)^2$

$(\frac{1}{2})^2, (\frac{3}{2})^2, (\frac{5}{2})^2, \ldots$

$\lambda_{-1} \quad \lambda_1 \quad \lambda_2$

Eigenfunctions:

$y_n = B\sin(\sqrt{\lambda_n}\,x)$

The shape of a jump rope being twirled satisfies the following equation:

$$
\begin{aligned}
Ty'' + \rho\omega^2 y &= 0 \\
y(0) &= 0 \\
y(L) &= 0
\end{aligned}
$$

ρ, T, L are constants.

Rearrange:

$$y'' + \left(\frac{\rho}{T}\omega^2\right) y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$y(x) = B\sin\left(\sqrt{\frac{\rho}{T}}\,\omega\, x\right)$$

Now $y(L) = B\sin\left(\sqrt{\frac{\rho}{T}}\,\omega\, L\right) = 0$

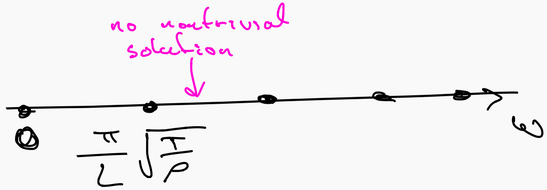
if $\sqrt{\frac{\rho}{T}}\,\omega L = n\pi$, B can be anything.

i.e. $\omega = n\frac{\pi}{L}\sqrt{\frac{T}{\rho}}$

angular frequency.

$$y = B\sin\left(\sqrt{\frac{\rho}{T}}\,\omega L\, x\right)$$

Otherwise: $B = 0$, $y = 0$



Conclusion: You can only rotate a jump rope at a certain angular frequencies:

Even though the equations

$$y'' + p(x)y' + \lambda y = 0, \quad y(0) = 0, \quad y'(0) = 0 \quad (1)$$

and

$$y'' + p(x)y' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \quad (2)$$

look superficially similar, the solutions have very different properties.

- For (1), existence and uniqueness of IVPs usually applies, no matter what λ is.

Depending on λ, only trivial

- For (2), sometime there is ~~no~~ solutions y, otherwise there are infinitely many. ~~depending on λ~~
    - Alternatively, we may think of (2) as a joint equation for λ,y.

    To solve (2), we have to solve for both λ and y.