MAT303: Calc IV with applications

Lecture 15 - March 28 2021

mass damping coef. spring constant ext force

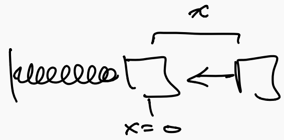
Recap:

- We are interpreting $my'' + cy' + ky = f(t)$

  as a mass-spring system.

- When $f(t) = 0$, saw that there are 3 regimes, depending on whether $c < 4km$.

- Last time, we saw how to solve the equation when $f(t)$ isnonzero.



$F = kx$

k big: stiff spring

Today:

- Physical interpretation of $my'' + cy' + ky = f(t)$ when $f(t)$ is nonzero (Ch 3.6)
  - Resonance for damped and undamped forced oscillations
  - Transient and steady periodic solutions

Recall: (Lecture 13)

$$mx'' + kx = 0$$

- m: mass
- k: The constant such that Force = $k \cdot$ displacement

describes unforced, undamped oscillations.

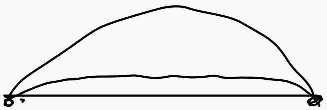
$= A\cos(\omega_0 t) + B\sin(\omega_0 t).$

Solution is: $x = C\cos(\omega_0 t - \alpha)$, where

- $\omega_0 = \sqrt{\frac{k}{m}}$ does not depend on initial conditions.
- C and α depend on initial conditions

Examples:

- Mass on spring
- Guitar/piano string
- Bridge swinging side to side/up down
- Child on swing (pendulum)
- Wine glass



Now assume we put in an external force of F_0:

$$mx'' + kx = F_0$$

Particular Solution:

$f(t) = F_0$
$f' = 0$
$f'' = 0$
$f''' = 0$

$A \cdot F_0 \approx A$

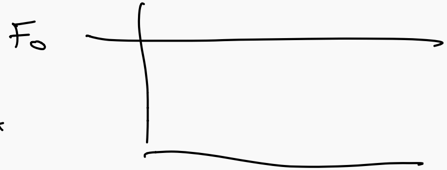
- Trial solution $x(t) = A$

$mx'' + kx = F_0$

$x = A$

$\Rightarrow 0 + kA = F_0$

$\Rightarrow A = \frac{F_0}{k}.$



- Solution is $x(t) = F_0/k$

Full soln is

$x = C\cos(\omega_0 t - \alpha) + \frac{F_0}{k}.$

Recall: (Lecture 13)

$$mx'' + kx = 0$$

- m: mass
- k: The constant such that Force $= k \cdot$ displacement

describes unforced, undamped oscillations.

Solution is: $x = C\cos(\omega_0 t - \alpha)$, where

- $\omega_0 = \sqrt{\dfrac{k}{m}}$
- C and α depend on initial conditions

Homog solns:

$x = \sin(\omega_0 t)$

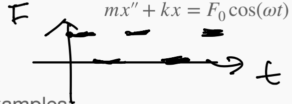
$x = \cos(\omega_0 t)$

Examples:

- Mass on spring
- Guitar/piano string
- Bridge swinging side to side/up down
- Child on swing (pendulum)
- Wine glass

Now assume external force is periodic:

$$mx'' + kx = F_0\cos(\omega t)$$



- F_0 : external force amplitude
- ω : (angular) frequency of ext. force

Examples:

- Mass on spring
- Guitar/piano string
- Bridge swinging side to side/up down
- Child on swing (pendulum)
- Wine glass

External force:

- Motor?
- Vibrations caused by other sounds
- Soldiers marching, wind blowing
- Adult pushing the swing
- Vibration caused by someone singing

Particular Solution:

$f(t) = F_0\cos(\omega t)$

$f'(t) = -\omega F_0 \sin(\omega t)$

Trial solution: $f''(t) = -\omega^2 F_0 \cos(\omega -$

$x = A\cos(\omega t) + B\sin(\omega t)$.

$x'' = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$

So $-Am\omega^2\cos(\omega t) + kA\cos(\omega t)$ ~~$+ mB\sin(\omega t) - Bk\omega^2\sin(\omega t)$~~ $= F_0\cos(\omega t)$

So $B = 0$, $A = \dfrac{F_0}{k - m\omega^2}$.

https://www.desmos.com/calculator/tpirvpcwbe

So $x_p(t) = \dfrac{F_0}{k - m\omega^2}\cos(\omega t)$.

If $\omega = \omega_0$

Trial solution is

$At\cos(\omega t)$

$+ Bt\sin(\omega t)$

Now assume external force is periodic:

$$mx'' + kx = F_0\cos(\omega t)$$

- m: mass
- k: The constant such that
  Force = $k \cdot$ displacement
- F : external force amplitude
- ω : (angular) frequency of ext. force

Examples:

- Mass on spring
- Guitar/piano string
- Bridge swinging side to side/up down
- Child on swing (pendulum)
- Wine glass

External force:

- Motor?
- Vibrations caused by other sounds
- Soldiers marching, wind blowing
- Adult pushing the swing
- Vibration caused by someone singing

$$x_p(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t).$$

$$= \frac{F_0/m}{\frac{k}{m} - \omega^2}\cos(\omega t).$$

Now $\omega_0 = \sqrt{\frac{k}{m}}$

Solution:

If $\omega \neq \omega_0$: $x(t) = C\cos\left(\omega_0 t - \alpha\right) + \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t$

If $\omega = \omega_0$: $x(t) = C\cos\left(\omega_0 t - \alpha\right) + \dfrac{F_0}{2m\omega_0}t\sin\omega_0 t$

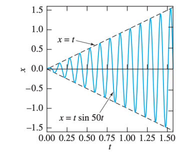


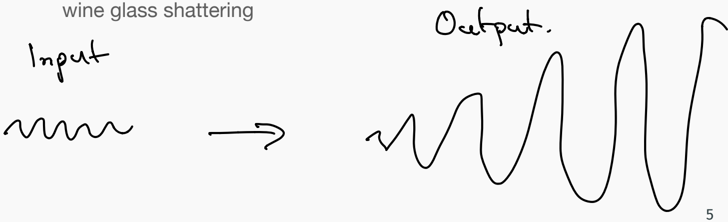
FIGURE 3.6.4. The phenomenon of resonance.

Observations

- New amplitude of x_p is much bigger than what if we just use $f(t) = F_0$
  - Amplitude is $\dfrac{F_0/k}{1 - (\omega/\omega_0)^2}$

    $$\frac{F_0/m}{\frac{k}{m} - \omega^2}\cdot\frac{\frac{m}{k}}{\frac{m}{k}} = \frac{F_0/k}{1 - \frac{\omega^2}{\omega_0^2}}$$

  - Call $\rho = \dfrac{1}{|1 - (\omega/\omega_0)^2|}$ the amplification factor.
  - As $\omega \to \omega_0$, this amplification goes to ∞ .
  - This is the phenomenon of **resonance**.
  - Roughly speaking: when external force is synchronized with natural frequency, amplitude gets very large.
  - Causes bridge collapse, sympathetic resonance (music), wine glass shattering



No external force:

$mx'' + cx + kx = 0$

- c : damping coefficient

Solution:

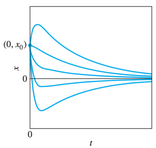


FIGURE 3.4.7. Overdamped motion: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

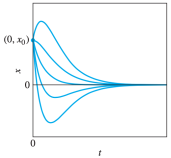


FIGURE 3.4.8. Critically damped motion: $x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

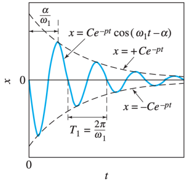


FIGURE 3.4.9. Underdamped oscillations: $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.

As long as there is damping, $c > 0$, the solutions go to 0.

The solutions are called **transient.**

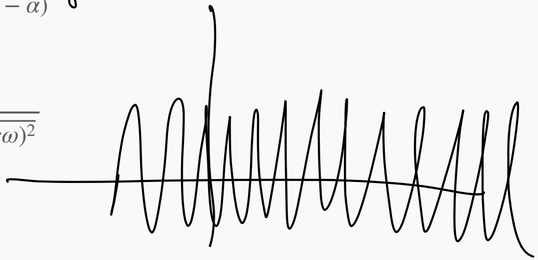
$$mx'' + cy + kx = F_0 \cos(\omega t)$$

- F_0 : external force amplitude
- ω : (angular) frequency of ext. force

Trial solution: $x(t) = A\cos(\omega t) + B\sin(\omega t)$ or $x(t) = C\cos(\omega t - \alpha)$

sp means steady period:

Solution: $x_{sp}(t) = C\cos(\omega t - \alpha)$



Where:

- $C = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$
- $\tan(\alpha) = \dfrac{c\omega}{k - m\omega^2}$

Observations:

- Amplitude is always finite, unlike undamped case
  - Amplitude close to F_0/k if ω is very small $\omega = 0 \Rightarrow C = \frac{F_0}{k}$.
  - Amplitude small if ω is very large $\omega \to \infty$, $C \to 0$.
  - Amplitude attains a maximum for some ω (minimize the denominator)

(Practical Resonance)

Full solution: $x = x_{tr} + x_{sp}$ In the long run, $x_{tr} \to 0$

$x \approx x_{sp}$.

https://www.desmos.com/calculator/o9js9ofznz

Initial conditions are forgotten.

Consider

$$mx'' + cy + kx = f(t)$$

$82 \cos 4t$

- $m = 1, c = 2, k = 26$
- External force $f(t) = 82\cos(4t)$
- $x(0) = 6, \quad x'(0) = 0$

Questions:
- Transient motion?
- Steady periodic oscillations?
- Practical resonance?

Homogeneous solution:

$$x'' + 2x' + 26x = 0$$

$$\Rightarrow r^2 + 2r + 26 = 0 \quad \text{(char eqn)}$$

$$r = -1 \pm \sqrt{1-26} = -1 \pm 5i$$

So

$$x(t) = A e^{-t}\cos(5t) + B e^{-t}\sin(5t)$$

Particular solution: (steady periodic soln).

Trial Soln:

$$x = 26A \cos 4t + 26 B \sin 4t$$

$$2x' = -8A \sin 4t + 8B \cos 4t$$

$$x'' = -16A \cos 4t - 16B \sin 4t$$

$$10A + 8B = 82$$
$$-8A + 10B = 0. \quad \Rightarrow \quad A = 5, \; B = 4.$$

So $x_{sp} = 5\cos 4t + 4 \sin 4t$

Full Soln:

~~Practical resonance?~~

$$x(t) = A e^{-t}\cos(5t) + B e^{-t}\sin(5t) + 5\cos 4t + 4\sin 4t$$

Note: as $t \to \infty$, $x \approx x_{sp}$. regardless of initial conditions.

ICs:

$A = 1, \; B = -3.$

Practical Resonance:

$$C = \frac{F_0.}{\sqrt{(26-\omega^2)^2 + (26\omega)^2}}$$

amplitude: Maximize C by minimizing $(26-\omega^2)^2 + 26\omega^2$

at $\omega = \sqrt{24}$.

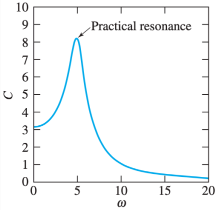


FIGURE 3.6.9. Plot of amplitude C versus external frequency ω.

Today:

- Physical interpretation of $my'' + cy' + ky = f(t)$ when $f(t)$ is nonzero (Ch 3.6)
  - Resonance for damped and undamped forced oscillations
  - Transient and steady periodic solutions

Midterm: up to and including Ch 3.8.

Next time:

- Ch 3.8 Endpoint problems

$$y'' + p(x)y' + q(x)y = 0; \quad y(a) = 0, \quad y(b) = 0.$$