MAT303: Calc IV with applications

Lecture 14 - March 24 2021

Last time:

mass

resistance

- Physical interpretation of

  spring constant

  $$my'' + cy' + ky = f(t)$$

  external force.

  in terms of mass-spring systems

- When $f(t) = 0$, saw that there are 3 regimes, depending on whether $c < 4km$:
  - Underdamped
  - Critically damped
  - Overdamped

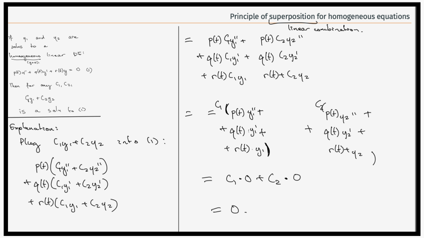
Today:
- Solving nonhomogeneous linear DEs

Principle of superposition for homogeneous linear equations:

If y_1 and y_2 are solutions to

$$ay'' + by' + cy = 0$$

Then $Ay_1 + By_2$ is also a solution.



Principle of $\underbrace{\text{superposition}}_{\text{linear combination.}}$ for homogeneous equations

If y_1 and y_2 are solns to a homogeneous linear DE:
(g=0).

$p(t)y'' + q(t)y' + r(t)y = 0 \quad (1)$

Then for any C_1, C_2,

$C_1y_1 + C_2y_2$

is a soln to (1)

Explanation:

Plug $C_1y_1 + C_2y_2$ into (1):

$p(t)(C_1y_1'' + C_2y_2'')$
$+ q(t)(C_1y_1' + C_2y_2')$
$+ r(t)(C_1y_1 + C_2y_2)$

$= p(t)C_1y_1'' + p(t)C_2y_2''$
$+ q(t)C_1y_1' + q(t)C_2y_2'$
$+ r(t)C_1y_1 \quad r(t)+C_2y_2$

$= C_1(p(t)y_1'' + q(t)y_1' + r(t)\cdot y_1) + C_2(p(t)y_2'' + q(t)y_2' + r(t)+y_2)$

$= C_1 \cdot 0 + C_2 \cdot 0$

$= 0.$

Using operator notation:

Let $Ly = ay'' + by' + cy$

if $Ly_1 = 0$ and $Ly_2 = 0$,

then

$$L(Ay_1 + By_2) = L(Ay_1) + L(By_2) = ALy_1 + BLy_2$$
$$= A \cdot 0 + B \cdot 0$$
$$= 0.$$

Principle of superposition for non-homogeneous linear equations:

If y_1 is a solution to

$$ay'' + by' + cy = f_1(x)$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(x)$$

Then $Ay_1 + By_2$ is a solution to

$$ay'' + by' + cy = Af_1(x) + Bf_2(x)$$

Proof: $Ly_1 = f_1$, $Ly_2 = f_2$

so

$$L(Ay_1 + By_2) = L(Ay_1) + L(By_2) = ALy_1 + BLy_2$$
$$= Af_1 + Bf_2.$$

How do we deal with external force, e.g.

$$y'' - 4y = 2e^{3x}\ ?$$

Recall (lecture 11)

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (2) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation in (3). If Y is any solution whatsoever of Eq. (2) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \tag{16}$$

for all x in I.

Roughly speaking:

complementary solution.

- All solutions are of the form $Y(x) = y_c + y_p$

where y_c is a solution to the homogeneous version of the equation.

Solution:

① Solution to homogeneous version:

$y'' - 4y = 0$

$\Rightarrow$ characteristic eqn is $r^2 - 4 = 0$

$r = \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

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② Find a particular solution:

Guess $y = Ae^{3x}$, sub into eqn. solve for A

$9Ae^{3x} - 4Ae^{3x} = 2e^{3x}$.

So $5A = 2$. So $A = 2/5$.

So $y_p = \frac{2}{5} e^{3x}$.

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③ General solution $y_c + y_p$:

$$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{2}{5} e^{3x}.$$

How do we deal with external force, e.g.

$$y'' + 3y' + 4y = 3x + 2?$$

Recall (lecture 11)

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for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_c + y_p$

  where y_c is a solution to the homogeneous version of the equation.

Solution:

① Solution to homogeneous version:

$$r^2 + 3r + 4 = 0 \Rightarrow r = \frac{-3 \pm \sqrt{9-16}}{2}$$

$$r = \frac{-3}{2} \pm \frac{\sqrt{7}i}{2}$$

$$y_c = e^{-3/2 x}\left(C_1 \cos \tfrac{\sqrt{7}}{2}x + C_2 \sin \tfrac{\sqrt{7}}{2}x\right).$$

② Find a particular solution:

Guess $y = Ax + B$, plug in, solve:

$$0 + 3A + 4Ax + 4B = 3x + 2$$

$$\Rightarrow 4A = 3 \Rightarrow A = 3/4 \Rightarrow \frac{9}{4} + 4B = 2 \Rightarrow B = \frac{-1}{16}.$$

③ General solution $y_c + y_p$:

$$y = e^{-3/2 t}\left(C_1 \cos \tfrac{\sqrt{7}}{2}t + C_2 \sin \tfrac{\sqrt{7}}{2}t\right) + \frac{4}{3}x - \frac{1}{4}$$

How do we deal with external force, e.g.

$$3y'' + 3y' - 2y = \overbrace{2\cos x}^{f}?$$

Take guess to be linear comb of f and all of its derivatives.

Recall (lecture 11)

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$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \qquad (16)$$

for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$ where y_h is a solution to the homogeneous version of the equation.

Solution:

① Solution to homogeneous version:

$y_c =$ __________

② Find a particular solution:

Guess $y = A\cos x$, plug in, solve:

$-3A\cos x - 3A\sin x - 2A\cos x = 2\cos x$

Doesn't work. No sin on RHS.

Instead guess $y = A\cos x + B\sin x$:

$-3A\cos x - 3A\sin x - 2A\cos x$

$-3B\sin x + 3B\cos x - 2B\sin x = 2\cos x$

i.e.

$-3A - 2A + 3B = 2$

$-3A - 3B - 2B = 0$

$\Rightarrow$ $-5A + 3B = 2$

$-3A - 5B = 0$

$$\Rightarrow \begin{pmatrix} -5 & 3 \\ -3 & -5 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

③ General solution $y_c + y_p$:

$y = y_c + A\cos x + B\sin x.$

How do we deal with external force, e.g.

$$y'' - 4y = 2e^{2x}?$$

Recall (lecture 11)

THEOREM 5 Solutions of Nonhomogeneous Equations

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$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \qquad (16)$$

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Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$

  where y_h is a solution to the homogeneous version of the equation.

Solution:

① Solution to homogeneous version:

$y'' - 4y = 0$

$\Rightarrow$ characteristic eqn is $r^2 - 4 = 0$

$r = \pm 2$

$y_c = C_1 e^{2x} + C_2 e^{-2x}$

② Find a particular solution:

Guess $y = Ae^{2x}$, sub into eqn. solve for A

$y'' - 4y = 4Ae^{2x} - 4Ae^{2x} = 0 \neq 2e^{2x}$

When guess is a homogeneous soln, it's the wrong guess.

Instead, guess $y = Axe^{2x}$, $y' = Ae^{2x} + 2Axe^{2x}$

$y'' - 4y = 2Ae^{2x} + 2Ae^{2x} + 4Ae^{2x}\cdot x - 4Ae^{2x}\cdot x = 2e^{2x}$

$= 4Ae^{2x} = 2e^{2x} \Rightarrow A = 1/2.$

③ General solution $y_c + y_p$:

$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{2} \cancel{x} e^{2x}.$

$ay'' + by' + cy = f$

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \ldots$ →

RULE 1 Method of Undetermined Coefficients

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

To solve constant coefficient linear differential equation $ay'' + by' + cy = f(x)$,

1. Find the homogeneous (i.e. complementary) solutions y_c
2. Check that f and its derivatives don't satisfy the homogeneous equation
3. Determine y_p by guessing $y_p =$ linear combination of f and its derivatives, and solve for coefficients
4. General solution is $y_c + y_p$

E.g.
$f = e^{kx}$
$f' = ke^{kx}$
$f'' = k^2 e^{kx}$
$\to y = e^{kx}$

E.x.
$f = \cos x$
$f' = -\sin x$
$f'' = -\cos x$
$\to y = A\cos x + B\sin x$

Ex.
$f = xe^x$
$f' = e^x + xe^x$
$f'' = e^x + e^x + xe^x$
$= 2e^x + xe^x$
$= 2(f' - f) + f$
$= 2f' - f.$
$\to y = Axe^x + Be^x.$

Examples:

- $y'' - 4y = 2e^{3x}$? $y_p = Ae^{3x}$
- $y'' + 3y' + 4y = 3x + 2$ $y = Ax + B.$
- $3y'' + 3y' - 2y = 2\cos x$ $y = A\cos x + B\sin x$
- $y'' - 4y = 2e^{2x}$ ✗ doesn't work.

✓

$y = A(3x + 2) + B3.$

$= 3Ax + \underline{2A + 3B}$

6:59.

How to deal with this situation?

$$y'' - 4y = 2e^{2x}.$$

Want to take $y_p = Ae^{2x}$ as trial solution, but it's a solution to the homogeneous equation.

Solution: Multiply trial solution by x.

$$y_c = Ae^{2x} + Be^{-2x}$$

In general:

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \dots$

RULE 2 Method of Undetermined Coefficients

If the function $f(x)$ is of either form in (14), take as the trial solution

$$\begin{aligned} y_p(x) = x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx \\ + (B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx], \end{aligned} \tag{15}$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (15) by substituting y_p into the nonhomogeneous equation.

Translation: keep multiplying trial solution by x until it's no longer a solution to the homogeneous equation.

Consider the equation

$$y'' + y = \tan(x).$$

Homogeneous solutions:

$$y = A\cos(x) + B\sin(x)$$

Unfortunately, the sequence $f, f', f'', \ldots$ has infinitely many linearly independent terms.

$$\sec^2 x, \quad 2\sec^2 x \tan x, \quad 4\sec^2 x \tan^2 x + 2\sec^4 x, \quad \ldots$$

I.e. The vector space spanned by f and its derivatives is infinite dimensional.

We can't use method of undetermined coefficients.

Variation of parameters

Want to solve $\underbrace{y'' + Py' + Qy}_{Ly} = f(x)$

Want $Ly = f$

Key ideas:

- Guess $y_p = u_1 y_1 + u_2 y_2$ where y_1, y_2 are the homogeneous solutions.

  Plug this into $L[y_p] = f(x)$ and see what this tells us about u_1, u_2 .

- We are free to make one more additional constraint to make our computation easier

(1) $y_p = u_1 y_1 + u_2 y_2$

prod. rule

(2) $\Rightarrow y_p' = (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2)$

Assume $(u_1' y_1 + u_2' y_2) = 0.$

(3) $y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$

(4) Since $y_i'' + P y_i' + Q y_i = 0$ For $i = 1,2$

so $y_i'' = -P y_i' - Q y_i$

Therefore.

$y_p'' + P y_p' + Q y_p = u_1' y_1' + u_2' y_2'$

i.e. $L[y_p] = \boxed{\begin{matrix} u_1' y_1' + u_2' y_2' = f \\ (u_1' y_1 + u_2' y_2) = 0. \end{matrix}}$

$\Rightarrow \begin{pmatrix} y_1' & y_2' \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$

$\rightarrow$

THEOREM 1 Variation of Parameters

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$\boxed{y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx,} \tag{33}$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

For our example:

$y_1 = \cos x$

$y_2 = \sin x$

$f(x) = \tan x.$

$y_p =$

See textbook, last ex.

Consider a system of two linear equations in two variables.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0; \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0.$$

$$\begin{pmatrix} y_1' & y_2' \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

Soln is

$$u_1' = \frac{f y_2}{y_1'y_2 - y_1y_2'}, \quad u_2' = \frac{-f y_1}{y_1'y_2 - y_1y_2'}$$

$$u_1' = \frac{-f y_2}{W(y_1, y_2)}, \quad u_2' = \frac{f y_1}{W(y_1, y_2)}$$

$$u_1 = \int \frac{-f y_2}{W(y_1, y_2)}\,dx \quad u_2 = \int \frac{f y_1}{W(y_1, y_2)}\,dx.$$

$$y = u_1y_1 + u_2y_2.$$

Back to our example:

$y'' + y = \tan(x).$

Homogeneous solutions:

$y_1 = \cos(x),\ y_2 = \sin(x)$

$$\underbrace{ay''+by'+cy}_{Ly} = f.$$

Why does method of undetermined coefficients work?

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \ldots$

RULE 1 Method of Undetermined Coefficients

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Need to explain why $y_p = C_0 f + C_1 f' + C_2 f'' + \cdots + C_n f^{(n)}$

Proof sketch:

1. Let V be the vector space spanned by $f, f', f'', f''', \ldots$.

   We are assuming that V is finite dimensional.

2. Then L is a linear operator $V \to V$.
3. By the rank nullity theorem from linear algebra,

   one of the two possibilities always holds:

   - $Lg = 0$ for some $g \in V$, i.e. $\dim(\ker L) > 0$.
   - $Lg = h$ always has a solution $g \in V$. any $h \in V$.

4. Therefore, if we assume that the first doesn't hold, then

   the second must hold.

What about rule 2?

RULE 2 Method of Undetermined Coefficients

If the function $f(x)$ is of either form in (14), take as the trial solution

$$\begin{aligned} y_p(x) = x^s[(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)e^{rx}\cos kx \\ + (B_0 + B_1 x + B_2 x^2 + \cdots + B_m x^m)e^{rx}\sin kx], \end{aligned} \tag{15}$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (15) by substituting y_p into the nonhomogeneous equation.

Need to explain why $y_p = x^s(C_0 f + C_1 f' + C_2 f'' + \cdots + C_n f^{(n)})$

Proof sketch:

1. Let V be the vector space spanned by $f, f', f'', f''', \ldots$.

   We are assuming that V is finite dimensional.

2. Then L is a linear operator $V \to V$, and $\dim(\ker L) > 0$.
3. Consider the vector space $W = \{x^s g : g \in V\}$.
   - Check: If s is small enough, $L : W \to V$.
   - Check: As s increases, $\dim(\ker L)$ decreases

4. Therefore, if s is just right, $\dim(\ker L) = 0$ and so $Lg = h$ always a solution

   $g \in W$.