MAT303: Calc IV with applications

Lecture 12 - March 17 2021

Recently:

- linear differential equations (Ch 3.1,3.2,3.3)
  - Homogeneous equations
  - Principle of superposition
  - Special case: constant coefficients
    - Different cases depending on number of real roots
  - Existence and uniqueness
  - Linear independence, and general solutions
- Tools:
  - Linear Differential Operators
  - Spend some more time on Euler's identity $e^{ix} = \cos(x) + i\sin(x)$

$$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \qquad (3)}$$

$$\boxed{a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t) \qquad (1)}$$

Today:

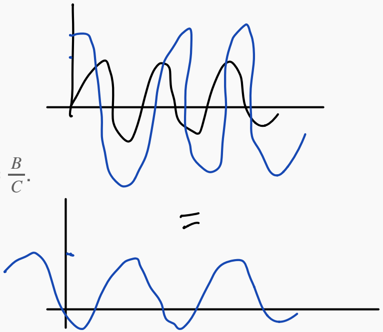
- Physical interpretation in terms of mass-spring systems

If $y = A\cos(\omega_0 t) + B\sin(\omega_0 t)$,

then $y = C\cos(\omega_0 t - \alpha)$, where

- $C = \sqrt{A^2 + B^2}$
- α satisfies $\cos\alpha = \frac{A}{C}$, and $\sin\alpha = \frac{B}{C}$.

$\Rightarrow \quad \tan\alpha = B/A$

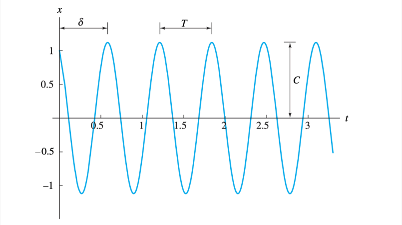


Terminology:

- ω_0 is the circular frequency
- α is the phase angle
- C is the amplitude

Also:

- $T = \dfrac{2\pi}{\omega_0}$ is the period
- $\nu = \dfrac{1}{T}$ is the frequency
- $\delta = \dfrac{\alpha}{\omega_0}$ is the time lag.



Proof:

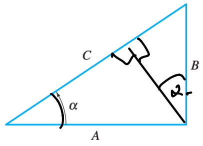


FIGURE 3.4.4. The angle α.

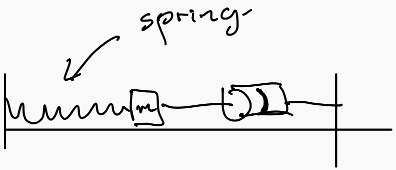
For $\omega_0 t = \alpha$:

Proof based on DEs:

if $y_2 = C\cos(\omega_0 t - \alpha)$,

$$y_2'' = -\omega_0^2 y_2$$

But this is the equation satisfied by $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ too.



ex: guitar string.

Assumptions:

- Total force has 3 parts:
- F_S spring restoring force $\propto$ displacement from equilibrium.
- F_R resistance $\propto$ $-$velocity.
- F_E external force. $F(t)$.
- $F = ma$

<u>Derivation:</u> Let $x(t)$ = displacement from equilibrium at time t.

Then $m\ddot{x} = F_S + F_R + F_E$

$$= -kx - c\dot{x} + F(t)$$

So $m\ddot{x} + c\dot{x} + kx = F(t)$

or $mx'' + cx' + kx = F(t)$.

Note: x and $\dot{x}$ may have different signs.

Example 1 A body with mass $m = \frac{1}{2}$ kilogram (kg) is attached to the end of a spring that is stretched 2 meters (m) by a force of 100 newtons (N). It is set in motion with initial position $x_0 = 1$ (m) and initial velocity $v_0 = -5$ (m/s). (Note that these initial conditions indicate that the body is displaced to the right and is moving to the left at time $t = 0$.) Find the position function of the body as well as the amplitude, frequency, period of oscillation, and time lag of its motion.

Hooke's Law:

$F = kd$

Here $100 = k2$, so $k = 50$.

$mx'' + cx' + kx = F(t)$.

$c = 0$, $F \equiv 0$, $m = \frac{1}{2}$, $k = 50$.

$\frac{1}{2}x'' + 50x = 0$.

Solution: Characteristic eqn:

$r^2 + 100 = 0$

$\Rightarrow r^2 = -100$

$r = \pm 10i$.

General solution is

$y = A\cos 10t + B\sin 10t$.

i.e. $y = C\cos(10t - \alpha)$.

$y' = -10C\sin(10t - \alpha)$

Initial conditions:

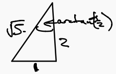
$y(0) = 1$, $y'(0) = -5$.

So $\begin{cases} C\cos(-\alpha) = C\cos(\alpha) = 1 \\ -10C\sin(-\alpha) = 10C\sin\alpha = -5. \end{cases}$ $C = \frac{1}{\cos\alpha}$.

So $10\tan(\alpha) = -5$, so $\tan\alpha = -\frac{1}{2}$.

$\alpha = \arctan(-\frac{1}{2})$.

So $C = \frac{1}{\cos(\arctan(-\frac{1}{2}))} = \frac{1}{\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}$.



Amplitude: $\frac{\sqrt{5}}{2}$.

Frequency: $\frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{5}{\pi}$

Time Lag: $\frac{\alpha}{\omega_0} = \frac{\arctan(-\frac{1}{2})}{10}$.

Period: $\frac{\pi}{5}$.

$y = C\cos(10t - \alpha$

Note: the frequency does not depend on initial conditions!!

Frequency is always $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Example 2 The mass and spring of Example 1 are now attached also to a dashpot that provides 1 N of resistance for each meter per second of velocity. The mass is set in motion with the same initial position $x(0) = 1$ and initial velocity $x'(0) = -5$ as in Example 1. Now find the position function of the mass, its new frequency and pseudoperiod of motion, its new time lag, and the times of its first four passages through the initial position $x = 0$.

$$\frac{1}{2}x'' + x' + 50x = 0$$

$$\Rightarrow x'' + 2x' + 100x = 0.$$

Characteristic eqn:

$$r^2 + 2r + 100 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 400}}{2}$$

$$= -1 \pm \sqrt{99}\, i$$

general solution is

$$x = Ae^{-t}\cos(\sqrt{99}t) + Be^{-t}\sin(\sqrt{99}t)$$

$$x = Ae^{-t}\cos(\sqrt{99}t) + Be^{-t}\sin(\sqrt{99}t)$$

$$x = e^{-t}\left(A\cos(\sqrt{99}t) + B\sin(\sqrt{99}t)\right)$$

$$x = Ce^{-t}\cos(\sqrt{99}t - \alpha)$$

First four passages

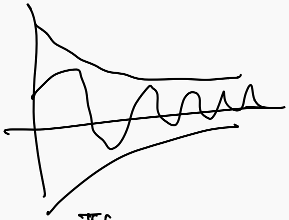
Solve $Ce^{-t}\cos(\sqrt{99}t - \alpha) = 0$,

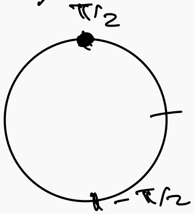
so $\cos(\sqrt{99}t - \alpha) = 0$

so $\sqrt{99}t - \alpha = \frac{\pi}{2} \pm l\pi$.

So $t = \dfrac{\alpha + \frac{\pi}{2} \pm l\pi}{\sqrt{99}}$

Can find first four.





Frequency: $\frac{1}{2\pi}\sqrt{99}$

Pseudoperiod: $\frac{2\pi}{\sqrt{99}}$.

Suppose you use a dashpot to stop a door from slamming shut. What is the best resistance for the dashpot?

Let's write down the solution to

$$mx'' + cx' + kx = 0$$

for arbitrary m, c, k.
Convenient to use:

$$x'' + 2px' + \omega_0^2 x = 0$$

$x'' + \underbrace{\frac{c}{m}}_{2p}x' + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0$

We know how to find all solutions using Ch 3.2, 3.3.

Characteristic eqn:

$r^2 + 2pr + \omega_0^2 = 0$

Roots

$r = -p \pm \sqrt{p^2 - \omega_0^2}$

3 cases:

large resistance.

if $p^2 > \omega_0^2$ (overdamped),

$\Leftrightarrow \left(\frac{c}{2m}\right)^2 > \frac{k}{m}$

$\Leftrightarrow c^2 > 4km.$

r_1, r_2 real.

negative

general solution is $x(t) = Ae^{r_1 t} + Be^{r_2 t}$

if $p^2 = \omega_0^2$ (Critically damped)

$(c^2 = 4km)$?

general solution is $x(t) = Ae^{-pt} + Bte^{-pt}$.

if $p^2 < \omega_0^2$ (Underdamped)

$c^2 < 4km.$

$r = -p \pm \sqrt{p^2 - \omega_0^2}$

$r = -p \pm \sqrt{\omega_0^2 - p^2}\, i$

general solution is

$x(t) = e^{-pt}\left(A\cos(\sqrt{\omega_0^2 - p^2}\, t) + B\sin(\sqrt{\omega_0^2 - p^2}\, t)\right)$

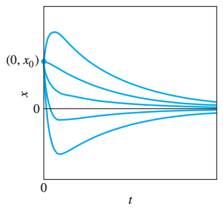


FIGURE 3.4.7. Overdamped motion: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with $r_1 < 0$ and $r_2 < 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

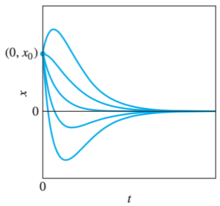


FIGURE 3.4.8. Critically damped motion: $x(t) = (c_1 + c_2 t)e^{-pt}$ with $p > 0$. Solution curves are graphed with the same initial position x_0 and different initial velocities.

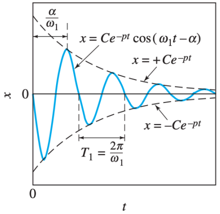


FIGURE 3.4.9. Underdamped oscillations: $x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)$.

$x(t) = Ae^{r_1 t} + Be^{r_2 t}$ $\quad r = -p \pm \sqrt{p^2 - \omega_0^2}$

Largest r controls rate of decay.

Largest coeff is $-p + \sqrt{p^2 - \omega_0^2}$

$$e^{-pt}\left(A\cos\left(\sqrt{\omega_0^2 - p^2}\, t\right) + B\sin\left(\sqrt{\omega_0^2 - p^2}\, t\right)\right)$$

$$p = \frac{c}{2m}$$

How do we deal with external force, e.g.

$$y'' - 4y = 2e^{3x} \; ?$$

Recall (lecture 11)

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (2) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation in (3). If Y is any solution whatsoever of Eq. (2) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \tag{16}$$

for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$

  where y_h is a solution to the homogeneous version of the equation.

Solution:

How do we deal with external force, e.g.

$$y'' + 3y' + 4y = 3x + 2?$$

Recall (lecture 11)

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  where y_h is a solution to the homogeneous version of the equation.

Solution:

Today:

- A simple example of how a higher order constant coefficient equation can arise from “the real world”.
- Terminology associated with simple harmonic motion
- The frequency of the solution does not depend on initial conditions!!
- The effect of the damping coefficient
- Started solutions of nonhomogeneous equations (Ch 3.5)