MAT303: Calc IV with applications

Lecture 12 - March 17 2021

Recently:

- Second order linear differential equations (Ch 3.1)
  - Homogeneous equations
  - Principle of superposition
  - Special case: constant coefficients
    - Different cases depending on number of real roots
  - Existence and uniqueness
  - Linear independence, and general solutions

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

Last time:

- Higher order linear differential equations (Ch 3.2)
- Mostly the same as second order linear differential equations
  - Difference: linear independence is more subtle
- Non-homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \quad (3)$$

Today:

- Constant coefficient higher order linear differential equations
- Similar to $n = 2$ case, but we will introduce some new tools:
  - Linear Differential Operators
- Spend some more time on Euler's identity $e^{ix} = \cos(x) + i\sin(x)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

$$y \to \boxed{L} \to$$

Consider the constant coefficient linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad (1)$$

We can rewrite this as

$$Ly = 0$$

a, D

where $L = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_0$ is an **operator**

where $D = \dfrac{d}{dx}$ is the **derivative operator**

Examples of operator notation:

operators, not numbers.

Fact: $(D-a)(D-b) = D^2 - (a+b)D + ab$

why?

$$\begin{aligned}(D-a)(D-b)f &= (D-a)(f' - bf)\\ &= D(f' - bf) - a(f' - bf)\\ &= f'' - bf' - af' + abf\\ &= f'' - (a+b)f' + abf\\ &= D^2 f - (a+b)Df + abf\\ &= (D^2 - (a+b)D + ab)f\end{aligned}$$

$y'' + 5y' + 6y = 0$

Example: Find a solution of the differential equation $(D^2 + 5D + 6)y = 0$.

We want $(D^2 + 5D + 6)y = 0$.

i.e. $(D+2)(D+3)y = 0$.

I could solve this just by solving $(D+3)y = 0$.

i.e. $y' + 3y = 0$

i.e. $y' = -3y$

So $y = e^{-3x}$ is a solution.

Example:

$$y - y' = x^2 + 3x$$

operator reformulation: $(1-D)y = x^2 + 3x$

$$y = \frac{1}{1-D}(x^2 + 3x)$$

geometric series

$$\begin{aligned}y &= (1 + D + D^2 + D^3 + \cdots)(x^2 + 3x)\\ &= x^2 + 3x + 2x + 3 + 2\\ &= x^2 + 5x + 5\end{aligned}$$

Verify: $y - y' = x^2 + 5x + 5 - (2x + 5)$
$= x^2 + 3x$ ✓

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$.

Let $p(x)$ be a polynomial of degree n. Then p has exactly n roots (possibly complex, possibly repeated).

Complex roots always appear in conjugate pairs $a + ib$ and $a - ib$

Example:

$x^2+1=0$, roots are $0+i$ and $0-i$

Consequence: Every polynomial of degree n can be written

$p(x) = (x-r_1) \cdots (x-r_n)$.

- It may be hard to find the roots (there is no "quadratic formula" for $n \geq 5$
- But we know by the theorem that they do exist

Let $i = \sqrt{-1}$, so $i^2 = -1$.

$i = i$
$i^2 = -1$
$i^3 = i^2 i = -i$
$i^4 = i^2 i^2 = (-1)(-1) = 1$
$i^5 = i^4 i = i$

Euler's identity:

$$e^{ix} = \cos x + i\sin(x)$$

Recall power series:

$i \to -1 \to -i \to 1$

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

E.g.

$$e^{i\pi} = \cos\pi + i\sin(\pi) = -1 + 0.$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} = \cos x + i\sin(x).$$

Let's try to solve

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

Use our only trick: guess $y = e^{rx}$.

Sub in $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$, ---

$y^{(n)} = r^n e^{rx}$.

get $a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0.$

factor out:

$e^{rx}(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) = 0.$

Factor polynomial:

$a_n e^{rx} (r - r_1) \cdots\cdots (r - r_n) = 0.$

So we get solutions

$y = e^{r_1 x}$, ---, $y = e^{r_n x}$.

If the $r_1, \ldots, r_n$ are distinct, the $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent. (Can compute Wronskian by induction).

So we have found n linearly independent solutions, so by the theory of last lecture, all solutions are of the form

$$y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$$

Product rule. $D(fg) = f\,Dg + g\,Df$.

Characteristic equation has repeated real roots

If the $r_1, \ldots, r_n$ not distinct, the $e^{r_1 x}, \ldots, e^{r_n x}$ are dependent.

By the theory of last lecture, we are missing some solutions.

We deduce the missing solutions by using linear differential operators.

Suppose we have 2 distinct roots, r_1 and r_2, where r_2 is repeated k times.

for simplicity

$$\underbrace{(a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0) = 0}_{\text{characteristic eqn.}}$$

$$a_n (r - r_1)(r - r_2)^k = 0$$

Original DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

Operator Formulation:

$$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D' + a_0)\, y = 0$$

$$\equiv \quad a_n (D - r_1)\underbrace{(D - r_2)^k y}\ = 0$$

Just solve $(D - r_2)^k y = 0$

~~Notice that~~ We already know $y = e^{r_2 x}$ is a soln to $(D - r_2)^k y = 0$

Try solns of the form $y = u e^{r_2 x}$.

function

Notice:

$$(D - r_2)\, u e^{r_2 x} = D(u e^{r_2 x}) - r_2 (u e^{r_2 x})$$

$$= (Du) e^{r_2 x} + u D e^{r_2 x} - r_2 (u e^{r_2 x})$$

$$= (Du) e^{r_2 x} + u r_2 e^{r_2 x} - r_2 (u e^{r_2 x})$$

$$= (Du) e^{r_2 x}.$$

Therefore

$$(D - r_2)^k (u e^{r_2 x}) = (D^k u) e^{r_2 x}$$

THEOREM 2 Repeated Roots

If the characteristic equation in (3) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation in (1) corresponding to r is of the form

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}. \quad (14)$$

If the $r_1, \ldots, r_n$ not distinct, the $e^{r_1 x}, \ldots, e^{r_n x}$ are dependent.

By the theory of last lecture, we are missing some solutions.

We deduce the missing solutions by using linear differential operators.

Suppose we have 2 distinct roots, r_1 and r_2, where r_2 is repeated k times.

Just solve $(D - r_2)^k y = 0$

Try solns of the form $y = u e^{r_2 x}$

Notice:

$$(D - r_2) u e^{r_2 x} = D(u e^{r_2 x}) - r_2 (u e^{r_2 x})$$
$$= (Du) e^{r_2 x} + u D e^{r_2 x} - r_2 (u e^{r_2 x})$$
$$= (Du) e^{r_2 x} + u r_2 e^{r_2 x} - r_2 (u e^{r_2 x})$$
$$= (Du) e^{r_2 x}.$$

Therefore

$$(D - r_2)^k (u e^{r_2 x}) = (D^k u) e^{r_2 x}$$

So we want. $(D^k u) e^{r_2 x} = 0$

Could take $u = (c_0 + c_1 x + c_2 x^2 + \cdots + c_{k-1} x^{k-1})$.

~~Notice that~~

In other words, we've found k different solutions to $(D - r_2)^k y = 0$, which are all solutions to $a_n (D - r_1)(D - r_2)^k y = 0$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad (1)$$

The k solutions are

$e^{r_2 x}, x e^{r_2 x}, \ldots, x^{k-1} e^{r_2 x}$.

(Wronskian $\Rightarrow$ linearly independent).

THEOREM 2 Repeated Roots

If the characteristic equation in (3) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation in (1) corresponding to r is of the form

e^{0x}

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}. \qquad (14)$$

$k = 3$

Example: Find a general solution of the fifth-order differential equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

Subbing in $y = e^{rx}$

$\downarrow$

Characteristic equation:

$$9r^5 - 6r^4 + r^3 = 0$$

$$\Rightarrow \quad r^3(9r^2 - 6r + 1) = 0$$

$$\Rightarrow \quad r^3(3r-1)^2 = 0.$$

roots: $0, 0, 0, \frac{1}{3}, \frac{1}{3}$.

So

$y = e^{0x} = 1$ is a soln.

$y = x \cdot 1$

$y = x^2$.

$y = e^{\frac{1}{3}x}$ is another soln

$y = xe^{\frac{1}{3}x}$ is another soln

General soln:

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{\frac{1}{3}x} + c_5 x e^{\frac{1}{3}x}.$$

$\cos(-y) = \cos(y)$
$\sin(-y) = -\sin(y)$

If the r_1 and r_2 are complex conjugate roots, $r_1, r_2 = a \pm ib$,

We get the two contributions $e^{r_1 x}$ and $e^{r_2 x}$ to the general solution.

We would prefer real solutions. We can get these by using Euler's formula.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

We have 2 solns:

$y_1 = e^{(a+ib)x}$, $y_2 = e^{(a-ib)x}$

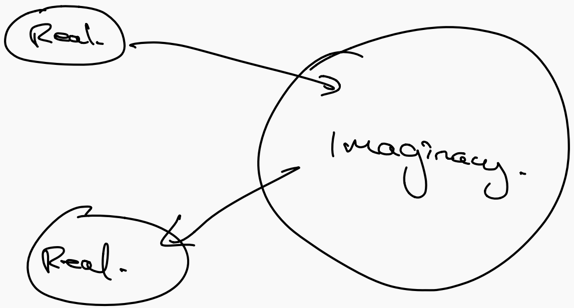
$y_1 = e^{ax} e^{ibx} = e^{ax}(\cos(bx) + i\sin(bx))$

$y_2 = e^{ax} e^{-ibx} = e^{ax}(\cos(bx) - i\sin(bx))$

So $\frac{1}{2}(y_1 + y_2) = e^{ax}\cos(bx)$ is a real soln

and $\frac{1}{2i}(y_1 - y_2) = e^{ax}\sin(bx)$ another soln.

Check: $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are independent.



THEOREM 3 Complex Roots

If the characteristic equation in (3) has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the corresponding part of a general solution of Eq. (1) has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx). \quad (21)$$

Example: Find a general solution of $y^{(4)} + 4y = 0$

Today:

- Constant coefficient higher order linear differential equations

$$\boxed{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad (1)}$$

  - Characteristic equation arises from substituting $y = e^{rx}$
    - If roots of characteristic equation are distinct, general solution is

$$y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$$

    - For a repeated root r of order k, the contribution to the general solution is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}$$

    - For a non-repeated conjugate pair of complex roots $a \pm bi$, the contribution to the general solution is

$$y = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

    - For repeated pairs of complex roots of order k, the contribution is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{ax} \cos(bx) + (d_1 + d_2 x + d_3 x^2 + \cdots + d_k x^{k-1}) e^{ax} \sin(bx)$$