MAT303: Calc IV with applications

Lecture 11 - March 15 2021

Recently:

- Second order linear differential equations (Ch 3.1)

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

  - Homogeneous equations
  - Principle of superposition
  - Special case: constant coefficients $ay'' + by' + cy = 0$
    - Different cases depending on number of real roots
  - Existence and uniqueness
  - Linear independence, and general solutions

Today:

- Higher order linear differential equations (Ch 3.2)
- Mostly the same as second order linear differential equations
  - Difference: linear independence is more subtle
- Non-homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \qquad (3)$$

Suppose we know that the 3rd order linear differential equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

has solutions $y_1(x) = e^{-3x}$, $\quad y_2(x) = \cos 2x$, $\quad y_3(x) = \sin 2x$.

$$y' = -3e^{-3x}$$

$$y'' = 9e^{-3x}$$

$$y^{(3)} = -27e^{-3x}$$

Recap: Principle of superposition for second order linear homogeneous DE

- For a second order linear differential equation

  $y'' + p(x)y' + q(x)y = 0$

  - If y_1 and y_2 are a pair of solutions, then $C_1y_1 + C_2y_2$ is another solution
  - Proof: Just plug $C_1y_1 + C_2y_2$ into the differential equation.

Principle of superposition for homogeneous equations

linear combination.

If y_1 and y_2 are solns to a homogeneous linear DE: $(g=0)$.

$p(t)y'' + q(t)y' + r(t)y = 0$ (1)

Then for any C_1, C_2,

$C_1y_1 + C_2y_2$

is a soln to (1)

Explanation:

Plug $C_1y_1 + C_2y_2$ into (1):

$$p(t)(C_1y_1'' + C_2y_2'') + q(t)(C_1y_1' + C_2y_2') + r(t)(C_1y_1 + C_2y_2)$$

$$= p(t)C_1y_1'' + p(t)C_2y_2'' + q(t)C_1y_1' + q(t)C_2y_2' + r(t)C_1y_1 + r(t)C_2y_2$$

$$= C_1(p(t)y_1'' + q(t)y_1' + r(t)y_1) + C_2(p(t)y_2'' + q(t)y_2' + r(t)y_2)$$

$$= C_1 \cdot 0 + C_2 \cdot 0$$

$$= 0.$$

Generalization: Principle of superposition for linear homogeneous DE

derivatives not exponents.

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \qquad (3)$$

THEOREM 1 Principle of Superposition for Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear equation in (3) on the interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \qquad (4)$$

is also a solution of Eq. (3) on I.

Proof: Plug in $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$.

Essentially the same as for $n=2$.

Suppose we know that the 3rd order linear differential equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

has solutions $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$.

1. By the **principle of superposition**, we know that we can get new solutions by taking linear combinations:
   $y(x) = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x$

- For a second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

  - If $p(x)$ and $q(x)$ and $f(x)$ are nice, for every choice of initial values $y(a) = \alpha$ and $y'(a) = \beta$,

    a solution will exist.

THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{8}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \tag{11}$$

Generalization: Existence and uniqueness for linear homogeneous DE

- If the coefficient functions are nice, then for every
- Nth order $\Longrightarrow$ need conditions on first derivatives $\underbrace{\text{0…n-1}}_{\text{n initial conditions.}}$ to specify solution completely.

THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions $p_1, p_2, \ldots, p_n$, and f are continuous on the open interval I containing the point a. Then, given n numbers $b_0, b_1, \ldots, b_{n-1}$, the nth-order linear equation (Eq. (2))

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the n initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}. \tag{5}$$

Suppose we know that the 3rd order linear differential equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

has solutions $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$.

1. By the **principle of superposition**, we know that we can get new solutions by taking linear combinations:
   $y(x) = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x$
2. By the **existence and uniqueness theorem** for initial value problems, we know that there are solutions satisfying any initial conditions
   E.g. the solution to
   $y(0) = 0, \quad y'(0) = 5, \quad y''(0) = -39$
   is $y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x.$

We have

$y(0) = -3 + 3 - 0 = 0.$

$y'(0) = 5$

$y''(0) = -39.$

Last time:

Linear independence of two functions:
Two functions are linearly independent if they are not multiples of each other

E.g. $y_1 = e^x$, $y_2 = 2e^x$, $y_3 = xe^x$ | y_1, y_2 dependent. because $y_2 = 2y_1$. y_1, y_3 are independent.

Linear independence of more than two functions:

DEFINITION Linear Dependence of Functions

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{7}$$

on I; that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x in I.

E.g. $2y_1 - y_2 = 0$, so y_1, y_2 dependent.

if $f_1, \ldots, f_n$ are dependent, then there is redundancy in the sense that f_1 is a linear combination of $f_2, \ldots, f_n$.

How to check that functions $f_1, \ldots, f_n$ are linearly independent?

Take the Wronskian:

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

if $W \equiv 0$, $f_1, \ldots, f_n$ are dependent
otherwise independent.

Recall Cofactor formula for determinants.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Suppose we know that the 3rd order linear differential equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0$$

has solutions $y_1(x) = e^{-3x}, \quad y_2(x) = \cos 2x, \quad y_3(x) = \sin 2x$.

1. By the **principle of superposition**, we know that we can get new solutions by taking linear combinations:
   $y(x) = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x$
2. By the **existence and uniqueness theorem** for initial value problems, we know that there are solutions satisfying any initial conditions, e.g. the solution to
   $y(0) = 0, \quad y'(0) = 5, \quad y''(0) = -39$
   is $y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x.$
3. The functions y_1, y_2, y_3 are linearly independent because the Wronskian is nonzero:

$$W = \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= e^{-3x}\begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} + 3e^{-3x}\begin{vmatrix} \cos 2x & \sin 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$+ 9e^{-3x}\begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 26e^{-3x} \neq 0.$$

$$= e^{-3x}\left(8\sin^2 2x + 8\cos^2 2x\right)$$

$$+ \;\; - \; - \; -$$

$$= e^{-3x}\, 8 + \cdots$$

- For a second order linear homogeneous differential equation
  - If y_1 and y_2 are a pair of linearly independent solutions, then every solution is of the form $C_1 y_1 + C_2 y_2$.
  - Contrast with the following statement which we already know:
    - If y_1 and y_2 are a pair of solutions, then $C_1 y_1 + C_2 y_2$ is another solution

Different.

Generalization: for the equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = 0. \tag{3}$$

- If $y_1, \ldots, y_n$ are linearly independent solutions, then all solutions are of the form $C_1 y_1 + \cdots + C_n y_n$.
- Contrast with the following statement which we already know:
  - If $y_1, \ldots, y_n$ are solutions, then $C_1 y_1 + \cdots + C_n y_n$ is another solution.

Technical statement (Ch 3.2):

THEOREM 4 General Solutions of Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = 0 \tag{3}$$

on an open interval I where the p_i are continuous. If Y is any solution whatsoever of Eq. (3), then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

for all x in I.

Suppose we know that the 3rd order linear differential equation

$$y^{(3)} + 3y'' + 4y' + 12y = 0 \qquad (1)$$

has solutions $y_1(x) = e^{-3x}$, $\quad y_2(x) = \cos 2x$, $\quad y_3(x) = \sin 2x$.

1. By the **principle of superposition**, we know that we can get new solutions by taking linear combinations:
   $y(x) = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin 2x$ (2)
2. By the **existence and uniqueness theorem** for initial value problems, we know that there are solutions satisfying any initial conditions, e.g. the solution to
   (*) $y(0) = 0, \quad y'(0) = 5, \quad y''(0) = -39$
   is $y(x) = -3e^{-3x} + 3\cos 2x - 2\sin 2x$.
3. The functions y_1, y_2, y_3 are **linearly independent** because the Wronskian is nonzero.
4. Therefore by **Theorem 4**, every solution of (1) is of the form (2).

To solve (*):

$$y'(x) = -3c_1 e^{-3x} - 2c_2 \sin 2x + 2c_3 \cos 2x$$

$$y''(x) = 9c_1 e^{-3x} - 4c_2 \cos 2x - 4c_3 \sin 2x$$

So constraints are

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0$$

$$y'(0) = 5 \Rightarrow -3c_1 + 2c_3 = 5$$

$$y''(0) = -39 \Rightarrow 9c_1 - 4c_2 = -39.$$

This is a linear system.

$$\begin{array}{c} y(0) \\ y'(0) \\ y''(0). \end{array} \begin{pmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 9 & -4 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -39 \end{pmatrix}$$

Could use gaussian elimination.

gives $c_1 = -3$, $c_2 = 3$, $c_3 = -2$.

"Wronskian matrix"

Recall from LA, if $\det \neq 0$, always exists solution to linear system.

Much of what we said only applies to homogeneous equations

- $$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \qquad (3)$$

What can we say about non-homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)?$$

Example: Consider

$$y^{(3)} + 3y'' + 4y' + 12y = 12x + 4 \qquad (*)$$

Suppose we know that $\underbrace{y = x}_{\text{particular soln.}}$ is a solution. Are there other solutions?

Cannot just take $y = c_1 x$.

Because

$$y^{(3)} + 3y'' + 4y' + 12y$$
$$= 0 + 0 + 4c_1 + 12c_1 x$$
$$= c_1(12x + 4). \quad \text{Not a solution.}$$

Recall: $y_h(x) = c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin(2x)$.

to $y^{(3)} + 3y'' + 4y' + 12y = 0$.

Take $y = y_h + x$, $y' = y_h' + 1$, $4y' = 4y_h' + 4$

Then

$$y^{(3)} + 3y'' + 4y' + 12y = y_h^{(3)} + 0 + 3y_h'' + 0 + 4y_h' + 4 + 12y_h + 12x$$
$$= 0 + 12x + 4$$

So $y = \underbrace{c_1 e^{-3x} + c_2 \cos 2x + c_3 \sin(2x)}_{y_h} + \underbrace{x}_{y_p}$.

is a soln to $(*)$.

Once we have one solution, we can generate more solutions by using the homogeneous solutions.

Are there are any more? No.

Much of what we said only applies to homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \quad (3)$$

What can we say about non-homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)? \quad (2)$$

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (2) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation in (3). If Y is any solution whatsoever of Eq. (2) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)}_{y_h} + y_p(x) \quad (16)$$

for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$ where y_h is a solution to the homogeneous version of the equation.
  y_p is any particular solution.

Proof: Suppose y_p is a solution to (2).
Let Y be another solution to (2).
Then plug in $y = Y - y_p$ into LHS of (2)
Then

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y$$

$$= Y^{(n)} + p_1(x)Y^{(n-1)} + \cdots + p_{n-1}(x)Y' + p_n(x)Y$$

$$-\left(y_p^{(n)} + p_1(x)y_p^{(n-1)} + \cdots + p_{n-1}(x)y_p' + p_n(x)y_p\right)$$

$$= f(x) - f(x) = 0.$$

So, $Y - y_p$ is a solution to homogeneous version.
So $Y - y_p = y_h$.
So $Y = y_h + y_p$.

Practical takeaway of all this:

- To find general solution of nonhomogeneous equation,
  - We only need to find a single solution y_p.
  - Combine this with the general solution to the homogeneous problem.
    - For the homogeneous problem, we only need to find n linearly independent solutions and then take their linear combination.