MAT303: Calc IV with applications

Lecture 10 - March 10 2021

Last time:

- Second order linear differential equations (Ch 3.1) $y'' + p(x)y' + q(x)y = r(x)$
  - Homogeneous equations $y'' + p(x)y' + q(x)y = 0$
  - Principle of superposition → if y_1 and y_2 are 2 solutions, Then $y = Cy_1 + Dy_2$ is a solution too.
  - Constant coefficient case
    - Real roots
    - Imaginary roots

$$y'' + ay' + by = 0$$

Key idea: Guess $y = e^{rt}$

Today:

- Second order linear differential equations (Ch 3.1)
  - Existence and uniqueness
  - Linear independence, and general solutions
    - Wronskian

- Consider the functions $y_1 = e^x$ and $y_2 = 3e^x$.

Then $Ay_1 + By_2 = Ce^x$.

$$Ae^x + 3Be^x = 7\,e^x.$$

$A=1 \quad B=2 \quad = 7e^x$

$A=4 \quad B=1$

Even though there are seemingly two parameters A and B, it is really a one parameter family.

- Contrast with the situation $y_1 = e^x$ and $y_2 = 3e^{2x}$.

Now $y = Ay_1 + By_2$ is genuinely a two-parameter family.

$$y = Ae^x + 3Be^{2x} \overset{x}{=} e^x\underbrace{(A + 3Be^x)}_{C}$$

Linear independence of functions:
Two functions are linearly independent
if they are not multiples of each other

i.e. $f = kg$

k constant.

Example

$3e^x,\ e^x$ are linearly dependent

because $3e^x = 3\cdot e^x$.

Example

$e^x,\ 3e^{2x}$ are linearly independent

because

$$ke^x \neq 3e^{2x}.$$

Last time: finding the general solution to

$$y'' + 5y' + 6y = 0$$

- Substituted in $y = e^{rt}$ as a guess
- Lead to the equation $r^2 + 5r + 6 = 0$
- Therefore $r = -2, -3$.
- So $y_1 = e^{-2t}$ and $y_2 = e^{-3t}$ are 'solutions'.
- By superposition, $y = Ae^{-2t} + Be^{-3t}$ is a solution too

  (for any choice of A and B)

Now suppose that we impose an initial condition $y(0) = 2$ and $y'(0) = 3$.

We have $y = Ae^{-2t} + Be^{-3t}$

Want $y(0) = A + B = 2$ (1)

$y'(t) = -2Ae^{-2t} - 3Be^{-3t}$

$y'(0) = -2A - 3B = 3$ (2)

So $2(1) + (2) \Rightarrow -B = 7 \Rightarrow B = -7$

So using (1), $A = 9$

So $y(t) = 9e^{-2t} - 7e^{-3t}$.

Things to notice:

- We needed 2 constraints to completely determine the solution.

Questions:

- Is this the only solution to the IVP?
- Why are there solutions at all? Will we always have solutions?
  - What happens if we change the initial conditions? $\leftarrow$

$y(0) = \alpha$
$y'(0) = \beta$

Then we want to solve.

$A + B = \alpha$ (1)

$-2A - 3B = \beta$ (2)

$$\equiv \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Solvable because $\det\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} = -5 \neq 0$.

Another motivating example:

$y'' - 2y' + y = 0$
$y(0) = 2$
$y'(0) = 3$

We see that if the characteristic has repeated roots we run into problems:

Motivation: How do we know that solutions to differential equations exist? How do we know that there's only one solution?

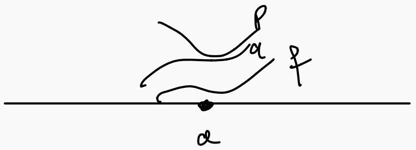
THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{8}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \tag{11}$$



$x^2 = -1$

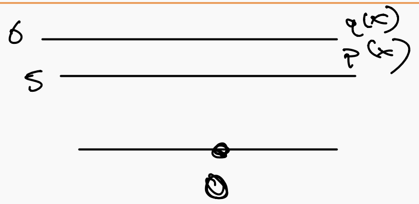
No real solution.

Previous example:

p(x) q(x) f(x)

$$y'' + 5y' + 6y = 0$$
$$y(0) = 2$$
$$y'(0) = 3$$

a b_0 b_1



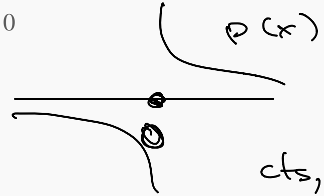
Conclude: solution exists, is unique.

An example where the theorem does not apply:

p(x)

$$y'' + x^{-1}y' + 6y = 0$$
$$y(0) = 2$$
$$y'(0) = 3$$

a=0.



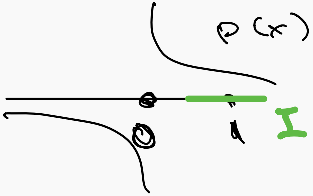
p(x) is not cts, so thm doesn't tell us anything.

An example where the theorem does apply:

p(x)

$$y'' + x^{-1}y' + 6y = 0$$
$$y(1) = 2$$
$$y'(1) = 3$$

a=1



Conclusion: There is a unique soln.

Linearly independent:

Not a multiple of each other.

Back to our example:

$y'' + 5y' + 6y = 0$

- By superposition, $y = Ae^{-2t} + Be^{-3t}$ is a solution (for any choice of A and B)

However, we still don't "know" that all the solutions are of the form $y = Ae^{-2t} + Be^{-3t}$.

For that, we need this theorem:

THEOREM 4 General Solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation (Eq. (9))

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq. (9) on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \qquad (8)$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \qquad (11)$$

$y = e^{-2t}$

$y = e^{-3t}$

We need to be able to check if e^{-2t} and e^{-3t} are linearly independent.

We see now that it is important to know whether two functions are linearly independent.

Here is an easy way to check if two functions are linearly independent.

THEOREM 3 Wronskians of Solutions

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation (Eq. (9))

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.

(a) If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.

(b) If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Wronskian:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Example: $y_1 = e^{-x}, \quad y_2 = xe^{-x}$

$y_1' = -e^{-x}$, $y_2' = e^{-x} - xe^{-x}$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-x}(e^{-x} - xe^{-x}) + e^{-x} x e^{-x}$$

$$= e^{-2x} - e^{-2x}x + e^{-2x}x$$

$$= e^{-2x} \neq 0.$$

never.

So y_1, y_2 are independent

Example: $y_1 = e^{-x}, \quad y_2 = 4e^{-x}$

$y_1' = -e^{-x}$ $y_2' = -4e^{-x}$

$$W(y_1,y_2) = -4e^{-2x} + 4e^{-2x} \equiv 0.$$

always. (no matter what x is).

So y_1, y_2 are linearly dependent.

continuous at a.

- For a second order linear differential equation homogeneous

  $y'' + p(x)y' + q(x)y = 0$

  - If $p(x)$ and $q(x)$ are nice, for every choice of initial values $y(a) = \alpha$ and $y'(a) = \beta$, a solution will exist, and it will be unique.
  - If y_1 and y_2 are a pair of linearly independent solutions, then every solution is of the form $C_1y_1 + C_2y_2$. (1)
  - We can check if y_1 and y_2 are linearly independent by computing the Wronskian. (2)

Example:

$y'' - 4y = 0$

$f(x)=0$

$p(x)=0$

$q(x)=-4$

We have two solutions $y_1 = e^{2x}$ and $y_2 = e^{-2x}$.

We also have solutions $w_1 = e^{2x} + e^{-2x}$ and $w_2 = e^{2x} - e^{-2x}$.

Conclusion: all solutions are of the form $C_1e^{-2x} + C_2e^{2x}$

different different.

Conclusion: all solutions are of the form $C_1(\underbrace{e^{-2x} + e^{2x}}_{w_1}) + C_2(\underbrace{e^{-2x} - e^{2x}}_{w_2})$

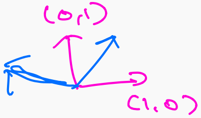
$y_1 = e^{2x}$ $\quad y_2 = e^{-2x}$

$y_1' = 2e^{2x}$ $\quad y_2' = -2e^{-2x}$

$W(y_1,y_2) = -2 - 2 = -4 \neq 0$

So by (2), y_1 y_2 are independent.

So by (1), all the solutions to $y''-4y=0$ are of the form $C_1e^{2x} + C_2e^{-2x}$.



$w_1 = e^{2x} + e^{-2x}$ $\quad w_2 = e^{2x} - e^{-2x}$

$w_1' = 2e^{2x} - 2e^{-2x}$ $\quad w_2' = 2e^{2x} + 2e^{-2x}$

$$W(w_1,w_2) = (2e^{4x} + 2e^{-4x} + 4) - (2e^{4x} + 2e^{-4x} - 4) = 8$$

So w_1, w_2 independent.

So

- For a second order linear differential equation

  $y'' + p(x)y' + q(x)y = 0$

  - If $p(x)$ and $q(x)$ are nice, for every choice of initial values $y(a) = \alpha$ and $y'(a) = \beta$, a solution will exist.
  - If y_1 and y_2 are a pair of linearly independent solutions, then every solution is of the form $C_1y_1 + C_2y_2$.
  - We can check if y_1 and y_2 are linearly independent by computing the Wronskian.

guess $y = e^{rt}$

$\Rightarrow (r^2 - 2r + 1) = 0$

$\Rightarrow (r-1)^2 = 0$

$\Rightarrow r = 1$

Example:

$y'' - 2y' + y = 0$

We have two solutions $y_1 = e^x$ and $y_2 = xe^x$.

Conclusion: all solutions are of the form $C_1e^x + C_2xe^x$

(This always happens when the characteristic equation has repeated roots)
See Theorem 6 in textbook.

Verify y_2:

$y_2' = e^x + xe^x$, $y_2'' = e^x + e^x + xe^x$
$= 2e^x + xe^x$

$y'' - 2y' + y = (2e^x + xe^x)$
$- 2(e^x + xe^x)$
$+ xe^x$
$= 0$

Also $W(e^x, xe^x) \neq 0$.
So e^x, xe^x are independent.

Consider the DE

$$ay'' + by' + cy = 0 \qquad (1)$$

Guess $y = e^{rt}$.

Then $y'' = r^2 e^{rt}$, $y' = r e^{rt}$

So by (1),

$$ar^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$$

$$\Rightarrow (ar^2 + br + c)e^{rt} = 0$$

$$\Rightarrow (ar^2 + br + c) = 0.$$

3 cases:

1) 2 distinct real solutions:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ are solutions

So $\boxed{y = Ae^{r_1 t} + Be^{r_2 t}}$ is the general solution.

$W(e^{r_1 t}, e^{r_2 t}) \neq 0$ ← check.

2) 1 real solution

$$r = \frac{-b}{2a}$$

Then $y_1 = e^{rt}$ and $y_2 = te^{rt}$ are solu.

So $\boxed{y = Ae^{rt} + Bte^{rt}}$ is the general soln.

3) 2 complex solutions.

$$r_1, r_2 = \underbrace{\frac{-b}{2a}}_{\mu} \pm \overbrace{\frac{\sqrt{b^2 - 4ac}}{2a}}^{i\nu} = \mu \pm i\nu$$

Then

$$\boxed{y = Ae^{\mu t}\cos(\nu t) + Be^{\mu t}\sin(\nu t)}$$

is the general solution.

Last time: finding the general solution to

$$y'' + 2y' + 2y = 0$$

- Substituted in $y = e^{rt}$ as a guess
- Lead to the equation $r^2 + 2r + 2 = 0$
- Therefore $r = -1 \pm i$.
- So $y_1 = e^{(-1+i)t}$ and $y_2 = e^{(-1-i)t}$ are ‘solutions’.