Problem 12

$\beta(t) = 4 \qquad \delta(t) = P \qquad h = 3$

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P - h$$

$$\boxed{\frac{dP}{dt} = (4 - P)P - 3}$$

Find CP

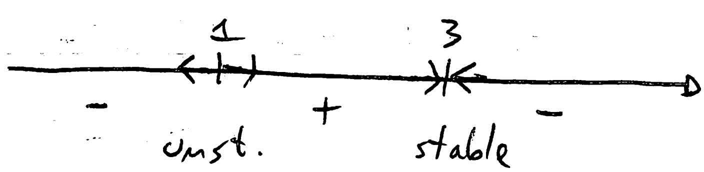
$(4 - P)P - 3 = 0$

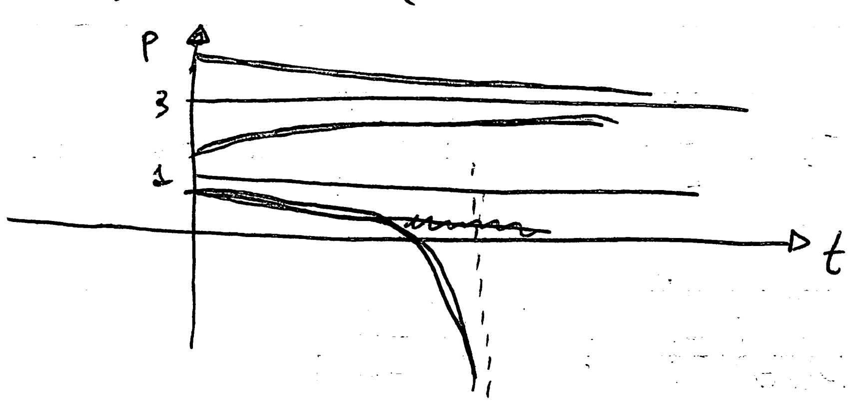
$-P^2 + 4P - 3 = 0$

$P^2 - 4P + 3 = 0$

$(P - 3)(P - 1) = 0$

$P = 3 \qquad P = 1$





Now we solve $\frac{dP}{dt} = (4 - P)P - 3$

$$= -(P-3)(P-1)$$

$$\frac{dP}{(P-3)(P-1)} = -\,dt$$

$$\int \frac{dP}{(P-3)(P-1)} = \int -\,dt = -t + C$$

$$\frac{A}{P-3} + \frac{B}{P-1} = \frac{AP - A + BP - 3B}{(P-3)(P-1)} = \frac{P(A+B) - A - 3B}{(P-3)(P-1)}$$

$$\begin{cases} A + B = 0 \\ -A - 3B = 1 \end{cases} \rightarrow \quad A = -B$$

$$B = \frac{-1 - A}{3} = \frac{-1 + B}{3}$$

$$3B = -1 + B \rightarrow \boxed{\begin{aligned} B &= -1/2 \\ A &= +1/2 \end{aligned}}$$

$$\int \frac{dP}{(P-3)(P-1)} = +\frac{1}{2}\int \frac{dP}{P-3} - \frac{1}{2}\int \frac{dP}{P-1} = +\frac{1}{2}\ln|P-3| - \frac{1}{2}\ln|P-1|$$

$$= \frac{1}{2}\ln\left|\frac{P-3}{P-1}\right|$$

Hence: $\frac{1}{2} \ln \left| \frac{P-3}{P-1} \right| = -t + C$

$$\ln \left| \frac{P-3}{P-1} \right| = -2t + 2C$$

$$\left| \frac{P-3}{P-1} \right| = e^{-2t} e^{2C}$$

$$\frac{P-3}{P-1} = \pm e^{-2t} e^{2C}$$

$$\frac{P-3}{P-1} = A e^{-2t} \qquad A = \pm e^{2C}$$

$$P-3 = (P-1) A e^{-2t} = AP e^{-2t} - A e^{-2t}$$

$$P(1 - A e^{-2t}) = 3 - A e^{-2t}$$

$$P = \frac{3 - A e^{-2t}}{1 - A e^{-2t}}$$

Initial condition: $P_0 = P(0) = \frac{3 - A}{1 - A}$

Solve for A:

$$(1-A) P_0 = 3 - A$$

$$A(1 - P_0) = 3 - P_0 \longrightarrow A = \frac{3 - P_0}{1 - P_0}$$

$$P(t) = \frac{3 - \frac{3-P_0}{1-P_0} e^{-2t}}{1 - \frac{3-P_0}{1-P_0} e^{-2t}} \cdot \frac{1-P_0}{1-P_0} = \frac{3(1-P_0) - (3-P_0) e^{-2t}}{(1-P_0) - (3-P_0) e^{-2t}}$$

Now we distinguish 3 cases:

CASE 1: $P_0 > 3$

Then $3 - P_0 < 0$ and $1 - P_0 < 0$

Hence $P(t) = \dfrac{3(1-P_0) - (3-P_0)e^{-2t}}{(1-P_0) - (3-P_0)e^{-2t}} =$

$$= \frac{-3(P_0-1) + (P_0-3)e^{-2t}}{-(P_0-1) + (P_0-3)e^{-2t}}$$

We check whether $P(t)$ explodes, namely if

$$-(P_0-1) + (P_0-3)e^{-2t} = 0 \quad \text{for some } t > 0$$

$$(P_0-3)e^{-2t} = (P_0-1)$$

$$e^{-2t} = \frac{P_0-1}{P_0-3}$$

$$-2t = \ln\left(\frac{P_0-1}{P_0-3}\right)$$

$$t = -\frac{1}{2} \underbrace{\ln\left(\frac{P_0-1}{P_0-3}\right)}_{\text{positive}}$$

Hence t is negative which impossible.

Hence $P(t)$ does not explode.

Hence $P(t)$ is a bounded population.

To find the limiting population, we solve

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{-3(P_0-1) + (P_0-3)\overset{\to 0}{e^{-2t}}}{-(P_0-1)+(P_0-3)\underset{\downarrow\, 0}{e^{-2t}}} = \frac{-3(P_0-1)}{-(P_0-1)} = \boxed{3}$$

<u>CASE 2</u> : <u>$1 < P_0 < 3$</u>

Also in this case $P(t)$ is bounded, and the limiting population is 3.

<u>CASE 3</u> : <u>$0 < P_0 < 1$</u>

Then $P_0 - 1 < 0$ and $P_0 - 3 < 0$

$$P(t) = \frac{3(1-P_0) - (3-P_0)e^{-2t}}{(1-P_0)-(3-P_0)e^{-2t}} = \quad \text{where } \begin{array}{l}(1-P_0) > 0\\ (3-P_0) > 0\end{array}$$

$$= \frac{(3-P_0)e^{-2t} - 3(1-P_0)}{(3-P_0)e^{-2t} - (1-P_0)}$$

We check whether $P(t)$ explodes, namely if $(3-P_0)e^{-2t} - (1-P_0) = 0$ for some $t > 0$.

$$e^{-2t} = \frac{1-P_0}{3-P_0} \quad \to \quad -2t = \ln\left(\frac{1-P_0}{3-P_0}\right)$$

$$t = -\frac{1}{2}\ln\left(\frac{1-P_0}{3-P_0}\right).$$

Now observe that

$$0 < \frac{1-P_0}{3-P_0} < 1 .$$

In fact $0 < \frac{1-P_0}{3-P_0}$ because $\begin{array}{l} 1-P_0 > 0 \\ 3-P_0 > 0 . \end{array}$

For the other inequality $\frac{1-P_0}{3-P_0} < 1$ we see that it is equivalent to:

$$1 - P_0 < 3 - P_0$$

$1 < 3$ which is true.

Then $\ln\left(\frac{1-P_0}{3-P_0}\right)$ is negative and

$t = -\frac{1}{2} \ln\left(\frac{1-P_0}{3-P_0}\right)$ is positive.

Hence $P(t)$ explodes when

$$t_{exp} = -\frac{1}{2} \ln\left(\frac{1-P_0}{3-P_0}\right).$$