1

Solutions Practice Final Exam

Problem 1

a

$$(1+x)\frac{dy}{dx} = 4y$$

$$\frac{dy}{dx} = \frac{4y}{1+x} = \frac{4\,y/x}{\frac{1}{x} + x/x} = \frac{4\,y/x}{\frac{1}{x}+1}.$$

Substitution: $v = y/x \qquad y = vx$

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

$$\frac{dv}{dx}x + v = \frac{4v}{\frac{1}{x}+1}$$

$$\frac{dv}{dx}x = \frac{4v}{\frac{1}{x}+1} - v = \frac{4vx}{1+x} - v = \frac{4vx - v - vx}{1+x} =$$

$$= \frac{3vx - v}{1+x} = \frac{3v(x-1)}{1+x}$$

Separable equation

$$\frac{1}{3v}\,dv = \frac{x-1}{x(x+1)}\,dx$$

Integrate:

$$\int \frac{1}{3v}\,dv = \int \frac{x-1}{x(x+1)}\,dx + C$$

$$\parallel$$

$$\frac{1}{3}\ln|v|$$

To solve this use partial fraction technique.

(2)

$$\int \frac{x-1}{x(x+1)}\,dx$$

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{x(A+B)+A}{x(x+1)}$$

$$\begin{cases} A+B = 1 \\ A = -1 \end{cases} \rightarrow \quad B = 1 - A = 1-(-1) = 2$$

$$\int \frac{x-1}{x(x+1)}\,dx = \int -\frac{1}{x}\,dx + \int \frac{2}{x+1}\,dx = -\ln|x| + 2\ln|x+1|$$

$$= -\ln|x| + \ln(x+1)^2$$

$$= \ln\left|\frac{(x+1)^2}{x}\right|$$

Hence:

$$\frac{1}{3}\ln|v| = \ln\left|\frac{(x+1)^2}{x}\right| + C$$

$$\ln|v| = 3\ln\left|\frac{(x+1)^2}{x}\right| + C$$

$$\ln|v| = \ln\left(\frac{(x+1)^6}{|x|^3}\right) + C$$

$$|v| = \frac{(x+1)^6}{|x|^3}e^C \qquad \text{But } v = Y/x$$

$$\boxed{Y = \pm x\frac{(x+1)^6}{|x|^3}e^C.}$$

(b)

$$x y' = 3y + x^3$$

$$x y' - 3y = x^3$$

$$y' - \frac{3}{x} y = x^2 \qquad \underline{\text{Linear}}$$

$$\rho(x) = e^{\int -3/x \, dx} = e^{-3 \ln|x|} = |x|^{-3} = x^{-3}$$

$$y' x^{-3} - \frac{3}{x} x^{-3} y = x^2 x^{-3}$$

$$\frac{d}{dx}(x^{-3} y) = x^{-1} \qquad \text{Integrate:}$$

$$x^{-3} y = \int x^{-1} \, dx + C$$

$$x^{-3} y = \ln|x| + C$$

$$\boxed{y = x^3 \ln|x| + C x^3}$$

c

$$x(x+y)\frac{dy}{dx} = y(x-y)$$

$$\frac{dy}{dx} = \frac{y(x-y)}{x(x+y)} = \frac{y}{x}\,\frac{x/x - y/x}{x/x + y/x} = \frac{y}{x}\,\frac{1-y/x}{1+y/x}$$ Homog.

Subst. $v = \frac{y}{x}$ $\quad y = vx \quad \frac{dy}{dx} = \frac{dv}{dx}x + v$

$$\frac{dv}{dx} + v = v\,\frac{1-v}{1+v}$$

$$\frac{dv}{dx} = v\,\frac{1-v}{1+v} - v = \frac{v - v^2 - \cancel{v} - v^2}{1+v} =$$

$$= \frac{-2v^2}{1+v}$$ Separable

$$\frac{1+v}{v^2}\,dv = -2\,dx$$ Integrate:

~~the~~ I solved this in class,
see your notes for the
last steps.

(d)

$$\frac{dy}{dx} = (4x + y)^2$$

subst. $v = 4x + y$

~~so~~ $y = v - 4x$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

$$\frac{dv}{dx} - 4 = v^2$$

$$\frac{dv}{dx} = v^2 + 4 \qquad \text{Separable}$$

$$\frac{dv}{v^2+4} = dx$$

$$\int \frac{dv}{v^2+4} = \int dx + C = x + C$$

$$\frac{1}{2} \operatorname{arctg}\left(\frac{v}{2}\right)$$

$$\operatorname{arctg}\left(\frac{v}{2}\right) = 2x + 2C$$

$$\frac{v}{2} = \tan(2x + 2C)$$

$v = 2\tan(2x + 2C)$ But $v = 4x + y$

$4x + y$

$$\boxed{y = 2\tan(2x + 2C) - 4x}$$

(e)

$$y^2 y' + 2x y^3 = 6x$$

$$\frac{dy}{dx} + 2xy = 6xy^{-2}$$ Bernoulli

$n = -2$

subst: $v = y^{1-n} = y^{1-(-2)} = y^3$

$$y = v^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3} v^{-2/3} \frac{dv}{dx}$$

$$\frac{1}{3} v^{-2/3} \frac{dv}{dx} + 2x v^{1/3} = 6x v^{-2/3}$$

Multiply by $v^{2/3}$

$$\frac{1}{3} \frac{dv}{dx} + 2xv = 6x$$

$$\frac{dv}{dx} + 6xv = 18x$$ Linear

$$\rho(x) = e^{\int 6x\,dx} = e^{3x^2}$$

$$\frac{dv}{dx} e^{3x^2} + 6x e^{3x^2} v = 18x e^{3x^2}$$

$$\frac{d}{dx}\left(e^{3x^2} v\right) = 18x e^{3x^2}$$ Integrate

$$e^{3x^2} v = \int 18x e^{3x^2}\,dx + C$$

$$e^{3x^2} v = 18\int x e^{3x^2} dx + C =$$

subst. $\left[\begin{array}{l} u = 3x^2 \\ du = 6x\,dx \end{array}\right]$

$$= 18 \frac{1}{6} \int 6x e^{3x^2} dx + C = 3\int e^u du + C =$$

$$= 3 e^u + C = 3 e^{3x^2} + C.$$

Hence:

$$v = 3 + \frac{C}{e^{3x^2}}$$ But $v = Y^3$

$$Y^3 = 3 + \frac{C}{e^{3x^2}}$$

$$\boxed{Y = \sqrt[3]{3 + \frac{C}{e^{3x^2}}}}$$

Problem 2

Tank of 400 gal

$V_0 = 100$ gal

Initial quantity of salt $= 50$ lb

$c_i = 1$

$r_i = 5$

$r_o = 3$

$x(t) =$ quantity of salt at time t

$$c_o = \frac{x}{V}$$

$$V = V_0 + (r_i - r_o)t = 100 + 2t$$

$$\frac{dx}{dt} = c_i r_i - c_o r_o = 5 - \frac{x}{100+2t}(3)$$

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5 \qquad \underline{\text{Linear}}$$

$$\rho(t) = e^{\int \frac{3}{100+2t}dt} = e^{3/2 \ln(100+2t)}$$

$$= (100+2t)^{3/2}$$

$$\frac{d}{dt}\left((100+2t)^{3/2}x\right) = 5(100+2t)^{3/2}$$

$$(100+2t)^{3/2}\,x = 5\int (100+2t)^{3/2}\,dt + C$$

$$= \frac{5}{2}\,\frac{(100+2t)^{3/2+1}}{\frac{3}{2}+1} + C$$

$$= \frac{5}{2}\,\frac{(100+2t)^{5/2}}{5/2} + C = (100+2t)^{5/2} + C.$$

Hence:

$$x(t) = \frac{(100+2t)^{5/2}}{(100+2t)^{3/2}} + \frac{C}{(100+2t)^{3/2}} =$$

$$= (100+2t) + \frac{C}{(100+2t)^{3/2}}$$

Initial condition: $x(0) = 50$

$$50 = x(0) = 100 + \frac{C}{100^{3/2}} = 100 + \frac{C}{1000}$$

$$50000 = 100000 + C$$

$$C = 100\,000 - 50\,000 = 50\,000$$

The tank is full when

$$V(t) = 100 + 2t = 400 \longrightarrow t = 150$$

Hence the quantity of salt when the tank is full is

$$x(150) = (100 + 300) + \frac{50\,000}{(100+300)^{3/2}} =$$

$$= 400 + \frac{50\,000}{400^{3/2}} =$$

$$= 400 + \frac{50\,000}{80\,000} = \boxed{400 + \frac{5}{8}}$$

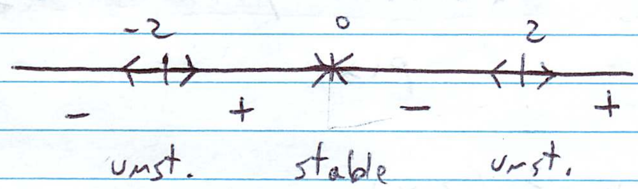
Problem 3

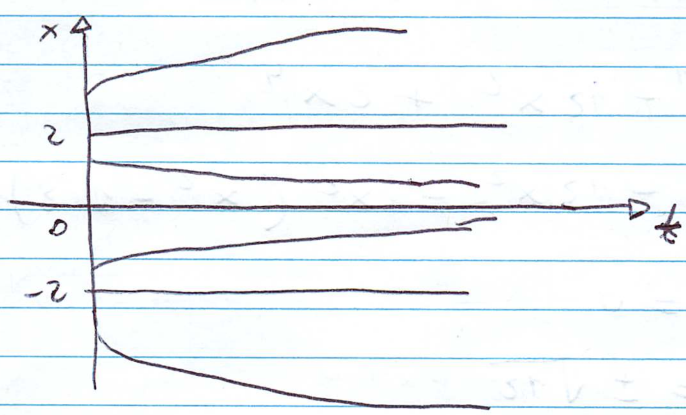
$$\frac{dy}{dx} = x^3(x^2-4) - h.$$

For $h=0$ find critical points and stability

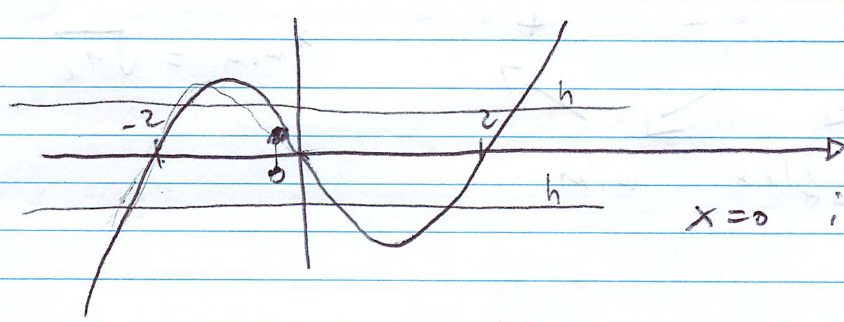
$$\frac{dy}{dx} = x^3(x^2-4)$$

$x^3(x^2-4)=0$ $\qquad x = 0, 2, -2$





Bifurcation: The graph of $x^3(x^2-4)$ is



$x=0$ is a flex.

Hence the number of solutions of

$$x^3(x^2-4) = +h$$

change when

$$h = p(x_{min}), \ p(x_{max})$$

where $p(x) = x^3(x^2-4)$

and x_{min}, x_{max} are the minimum and maximum points of $p(x)$.

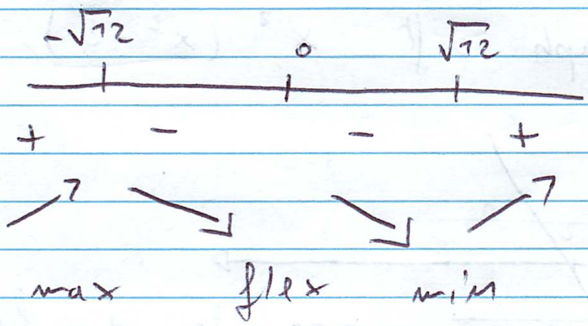
$$p'(x) = 3x^2(x^2-4) + x^3(2x)$$

$$= 3x^4 - 12x^2 + 2x^4$$

$$= x^4 - 12x^2 = x^2(x^2-12) = 0$$

$$x = 0$$

$$x = \pm\sqrt{12}$$



$x_{max} = -\sqrt{12}$

$x_{min} = \sqrt{12}$

$$h = p(-\sqrt{12}) = (-\sqrt{12})^3((-\sqrt{12})^2 - 4)$$

$$h = p(\sqrt{12}) = (\sqrt{12})^3((\sqrt{12})^2 - 4).$$

Problem 4

(a) $$y^{(iv)} + 18y'' + 81y = 0$$

$$r^4 + 18r^2 + 81 = 0$$

$$(r^2 + 9)^2 = 0$$

$r = \pm 3i$ each of multiplicity 2

$$y(x) = e^{0x}(c_1 \cos(3t) + c_2 \sin(3t)) +$$

$$+ x e^{0x}(c_3 \cos(3t) + c_4 \sin(3t))$$

$$= c_1 \cos(3t) + c_2 \sin(3t) + c_3 x \cos(3t) +$$

$$+ c_4 x \sin(3t).$$

(b)

$$y'' + y' + y = \sin^2 x$$

$y(x) = y_c + y_p$

Find y_c

$$r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{3}{2} i$$

$$y_c(x) = e^{-1/2 x} \left(c_1 \cos\left(\frac{3}{2} x\right) + c_2 \sin\left(\frac{3}{2} x\right)\right)$$

Find y_p

The derivatives of $\sin^2 x$ are

$2 \sin x \cos x$, $2 \cos^2 x - 2 \sin^2 x$, ...

Hence they are linear combinations of

$\sin^2 x$, $\sin x \cos x$, $\cos^2 x$.

Our guess is then

$$y_p(x) = A \sin^2 x + B \sin x \cos x + C \cos^2 x.$$

$$Y_p'(x) = 2A \sin x \cos x + B\cos^2 x - B\sin^2 x$$
$$-2C \cos x \sin x$$

$$= (2A - 2C)\sin x \cos x + B\cos^2 x - B\sin^2 x$$

$$Y_p''(x) = (2A-2C)\cos^2 x - (2A-2C)\sin^2 x$$
$$-2B\cos x \sin x - 2B \sin x \cos x$$

$$= (2A-2C)\cos^2 x + (2C-2A)\sin^2 x$$
$$-4B \sin x \cos x.$$

Plug-in:

$$(2A-2C)\cos^2 x + (2C-2A)\sin^2 x - 4B\sin x \cos x$$
$$+ (2A-2C)\sin x \cos x + B\cos^2 x - B\sin^2 x$$
$$+ A\sin^2 x + B\sin x \cos x + C\cos^2 x$$
$$=$$
$$\sin^2 x$$

Group factors on the LHS

$$\cos^2(x)\left(2A - 2C + B + C\right) + \sin^2 x\left(2C - 2A - B + A\right) +$$
$$+ \sin x \cos x\left(-4B + 2A - 2C + B\right) = \sin^2 x$$

Hence:

$$\begin{cases} 2A - C + B = 0 \\ 2C - A - B = 1 \\ 2A - 2C - 3B = 0 \end{cases}$$

Solve:

$R1 - R3 \longrightarrow C + 4B = 0 \longrightarrow C = -4B$

$$\begin{cases} 2A + 5B = 0 \\ -9B - A = 1 \end{cases} \longrightarrow A = -5/2\, B$$

$\longrightarrow 9B + A = -1$

$\downarrow$

$9B - 5/2\, B = -1$

$\downarrow$

$\frac{13}{2} B = -1 \longrightarrow B = -2/13$

$A = -5/2\,(-2/13) = 5/13$

$C = -4B = -4(-2/13) = 8/13$

(c)

$$y''' - y = e^x + 7$$

$$y = y_c + y_p$$

Find y_c

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0 \qquad r = 0, 1, -1 \quad \text{mult. } 1.$$

$$y_c(x) = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$$

$$= c_1 + c_2 e^x + c_3 e^{-x}$$

Find y_p

Guess: $y_p = A e^x + B$

But this won't work because

it is ~~[illegible]~~ of the same type

of y_c. (take $c_1 = B$, $c_2 = A$, $c_3 = 0$).

Another guess: $y_p(x) = A x e^x + B x$

$$y_p' = A e^x + A x e^x + B$$

$$y_p'' = A e^x + A e^x + A x e^x$$

$$= 2A e^x + A x e^x$$

$$y_p''' = 2Ae^x + Ae^x + Axe^x$$
$$= 3Ae^x + Axe^x$$

Plug-in

$$3Ae^x + \cancel{Axe^x} - \cancel{Axe^x} - Bx = e^x + 7$$

$$\begin{cases} 3A = 1 \\ -B = 7 \end{cases} \rightarrow \boxed{\begin{matrix} A = 1/3 \\ B = -7 \end{matrix}}$$