MAT303: Calc IV with applications

Lecture 6 - February 22 2021

Last time:

- Ch 1.6 Substitution methods:
  - $\frac{dy}{dx} = F(ax + by + c)$
  - $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$
  - $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Today:

- A few more substitutions (Reducible)
- Ch 2.1 (general) population models
  - $\frac{dP}{dt} = (\beta - \delta)P$
- Ch 2.2 Equilibrium solutions

By changing the functions β and δ we can model a large variety of phenomena.

Example: Solve $yy'' = \left(y'\right)^2$ for y.

Substitution:

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

This substitution works whenever you want to solve an equation of the form $F\left(y, y', y''\right) = 0$.

Example: Solve $yy'' = \left(y'\right)^2$ for y.

Substitution:

$$p = y' = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

This substitution works whenever you want to solve an equation of the form $F\left(y, y', y''\right) = 0.$

There is another useful substitution when $F\left(x, y', y''\right) = 0.$, see textbook Ch1.6.

2.1: Population models

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume

- Death rate is 0
- Birth rate is constant

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Example 1 Suppose that an alligator population numbers 100 initially, and that its death rate is $\delta = 0$ (so none of the alligators is dying). If the birth rate is $\beta = (0.0005)P$—and thus increases as the population does—then Eq. (1) gives the initial value problem

Assume
- Death rate is 0
- Birth rate is proportional to population

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume

- Constant death rate
- Birth rate is linear and decreasing with respect to population.

Interactive Graph: https://www.desmos.com/calculator/iw4c9k5fv1

Notice:

- No matter what initial condition, the limiting population is

$$\lim_{t\to\infty} P(t) = 150.$$

General population model:

$\frac{dP}{dt} = (\beta - \delta)P$

Assume

- Constant death rate
- Birth rate is linear and decreasing with respect to population.

$\frac{dP}{dt} = kP(M - P)$

Solution (next HW):

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{kMt}}$$

Notice:

- No matter what initial condition, the limiting population exists
- Limit population is $\lim_{t\to\infty} P(t) = M$.

1. Limited environment:
   - Suppose environment can only support up to M individuals.
   - Growth rate $\beta - \delta$ is proportional to $M - P$.
   - Then $\frac{dP}{dt} = (\beta - \delta)P = kP(M - P)$.
2. Competition situation:
   - Suppose birth rate constant, but deaths result from chance encounters between the members of the population.
   - Then $\frac{dP}{dt} = (\beta - \alpha P)P = kP(M - P)$.
3. Suppose we are modeling disease spread in a constant size population M.
   - This time, $P(t)$ represents the number of infected individuals.
   - Reasonable to assume that rate of new infections is proportional to $P(M - P)$.
   - Thus $\frac{dP}{dt} = kP(M - P)$ again.

The same equation can have multiple interpretations.

Different models can give rise to the same equation.

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume

- Death rate is constant
- Birth rate is proportional to time

Sometimes useful for modeling certain endangered species.

General population model:

$$\frac{dP}{dt} = (\beta - \delta)P$$

Assume
- Death rate is constant
- Birth rate is proportional to time

Sometimes useful for modeling certain endangered species.

Today:

- Ch 2.1 (general) population models
    - $\frac{dP}{dt} = (\beta - \delta)P$
- Ch 2.2 Equilibrium solutions

By changing the functions β and δ we can model and observe a large variety of phenomena:

- Exponential growth
- Bounded growth (logistic equation)
- Doomsday vs Extinction thresholds

2.2: Equilibrium solutions and stability

Example: Let k be a constant. What can we say about the solution to

$$\frac{dx}{dt} = -k(x - A)$$

Solution:

$$x(t) = A + \left(x_0 - A\right) e^{-kt}$$

Conclusion:

- As $t \to \infty$, temperature x approaches A.

Can get the same conclusion by looking at the phase diagram:

Example: Let k be a constant. What can we say about the solution to

$$\frac{dx}{dt} = kx(M - x)$$

Draw the phase diagram:

Conclusion:

First order autonomous:

$$\frac{dP}{dt} = f(P)$$

- “Update rule” only depends on current state (not time!)
- Long term behavior is easy to figure out.