MAT303: Calc IV with applications

Lecture 5 - February 17 2021

Today:

- Ch 1.6 Exact differential equations and

  Substitution methods

- Changing variables is a common operation
  - Can make the equation solvable/simpler
  - Can give more insight into the DE
  - Can demonstrate similarities between DEs.

Consider a function $f(x, y)$ of two variables.

$\dfrac{\partial}{\partial x}$ means “differentiate with respect to y”

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Example: $f(x, y) = x^2 + \sin(x)$

Then $\dfrac{\partial}{\partial y} f =$

And $\dfrac{\partial}{\partial x} f =$

More notation:

f_x means $\dfrac{\partial}{\partial x} f$

f_y means $\dfrac{\partial}{\partial y} f$

Higher order derivatives:

$\dfrac{\partial}{\partial x} \dfrac{\partial}{\partial y} f =$

$\dfrac{\partial}{\partial x} \dfrac{\partial}{\partial y} f =$

$\dfrac{\partial}{\partial y} \dfrac{\partial}{\partial x} f =$

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Notice:

$$f_{xy} = f_{yx}$$

This is true in general. It’s called Clairaut’s theorem.

One variable:

If $y = f(g)$

$$\frac{dy}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

If $y = \beta(x, v)$

$$\frac{dy}{dx} = \frac{\partial\beta}{\partial x} + \frac{\partial\beta}{\partial v}\frac{\partial v}{\partial x}$$

If $y = g(w, z)$

$$\frac{dy}{dx} = g_w w_x + g_z z_x$$

If $y = g(x, y, z)$

$$\frac{dy}{dt} = g_x x_t + g_y y_t + g_z z_t$$

Example:

$$\frac{d}{dx} x \sin(x)^2 =$$

Example:

$$f(x, y) = x^y, \quad y = y(x)\,.$$

What is f_x?

Don't try to memorize these exact formulas. Just remember the meaning and reasoning.

Example: Solve

$$\frac{dy}{dx} = (x + y + 3)^2$$

Using the substitution $v = x + y + 3$.

This works whenever you have a differential equation of the form

$$\frac{dy}{dx} = F(ax + by + c)\,.$$

Transforming a differential equation like this is very common in applications.

For homogeneous equations like $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$,

the substitution

$$v = \frac{y}{x}$$

is useful.

Example: Solve for y:

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

For the Bernoulli equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

You should take $v = y^{1-n}$ which turns it into

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)\,.$$

Example:

$$2xyy' = 4x^2 + 3y^2$$

Example:

$$2x + y^2 + 2xyy' = 0$$

Definition: A DE $M + N\dfrac{dy}{dx} = 0$ is **exact** if we can find $f(x, y)$ such that $f_x = M$ and $f_y = N$.

Is the following DE exact or not?

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Test for exactness: A DE $M + N\dfrac{dy}{dx} = 0$ can only be exact if $M_y = N_x$.

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Test for exactness: A DE $M + N\dfrac{dy}{dx} = 0$ can only be exact if $M_y = N_x$.

If we know an equation is exact, it is easy to figure out what f is.

$$\left(6xy - y^3\right) dx + \left(4y + 3x^2 - 3xy^2\right) dy = 0$$

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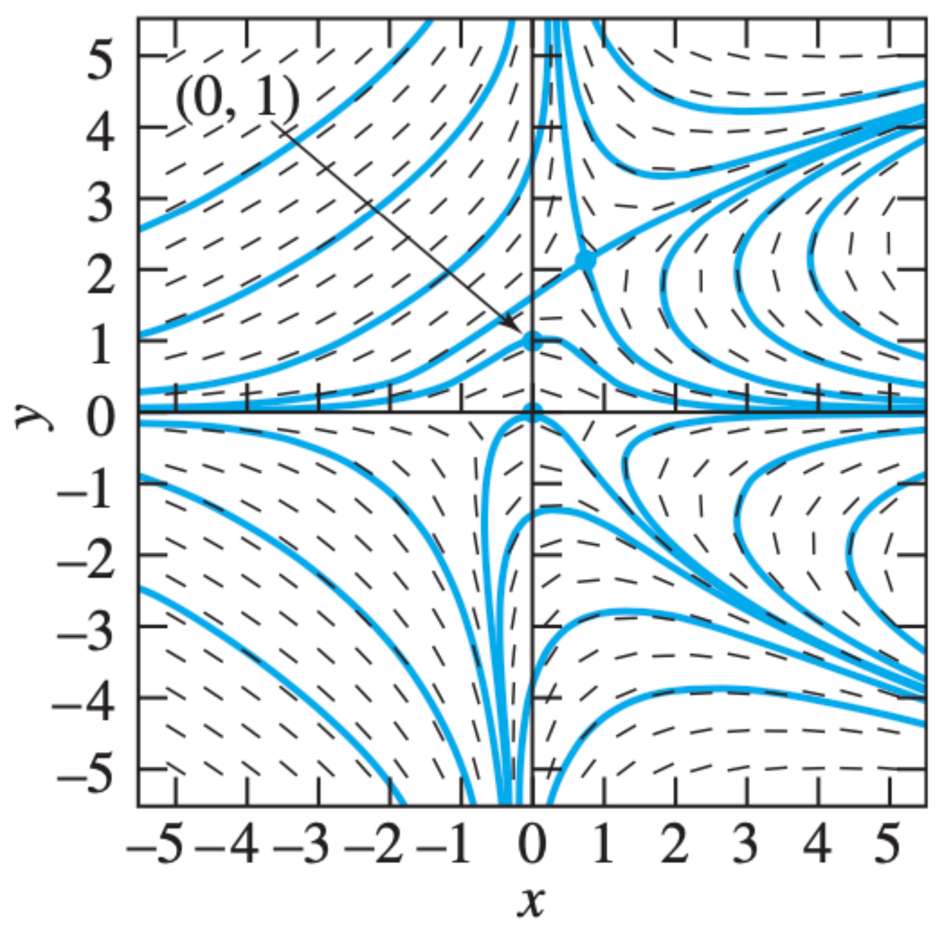

FIGURE 1.6.7. Slope field and solution curves for the exact equation in Example 9.

- Differential equations can often be transformed into different forms
  - Useful to reduce one problem to another problem
  - Useful when using a computer to solve the differential equation

- Exact equations are a nice application of Clairaut's theorem.