MAT303: Calc IV with applications

Lecture 4 - February 15 2021

Last time:

- Ch 1.4 Separable equations
    - $\dfrac{dy}{dx} = f(x)g(y)$
- Solving problems using separable equations. (Applications)
    - Radioactive decay
    - Water escaping from a tank

Today:

- Ch 1.5 Integrating Factors for first order linear Des

Types of DEs (these are not exclusive):

- First order: $\dfrac{dy}{dt} = f(x, y)$
- First order Separable: $\dfrac{dy}{dt} = f(x)g(y)$
- First order linear: $\dfrac{dy}{dx} + p(x)y = q(x)$

Classify:

$$\frac{dy}{dt} + 2y = 4t$$

$$y' = xy$$

$$y' = y^2 + x$$

$$\frac{dy}{dt} = 4t - 2y$$

$$y'' + 2y' + 3y = 0$$

$$(4 + t^2)\frac{dy}{dt} + 2ty = 4t$$

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- Ch 1.5 Integrating Factors

Separation of variables does not work:

$$(4 + t^2)\frac{dy}{dt} + 2ty = 4t$$

Product rule:

$$\frac{d}{dt}(fg) = f'g + fg'$$

E.g:

$$\frac{d}{dt}(t \sin t) =$$

$$\frac{d}{dt}(t^2 y) =$$

If $\dfrac{dy}{dt} = t^2$, what is y?

If $\dfrac{d}{dt}(yt) = t^3$, what is y?

If $\dfrac{d}{dt}((4 + t^2)y) = t$, what is y?

Example: Solve $(4+t^2)\dfrac{dy}{dt} + 2ty = 4t$

Key idea: writing LHS as $\dfrac{d}{dt}(\ldots)$

From previous slides:

$$\frac{d}{dt}((4+t^2)y) = (4+t^2)\frac{dy}{dt} + 2ty$$

If $\dfrac{d}{dt}((4+t^2)y) = 4t$, then $y = \dfrac{2t^2}{(4+t^2)}$

Example 1: Find the solution to the DE

$$\frac{dy}{dt} - 2y = e^{5t}$$

Want to use previous trick: LHS = $\dfrac{d}{dt}(\ldots)$?

How did we know to multiply by $\mu = e^{-2t}$?

Answer: $\mu = e^{\int p\,dt}$ always works.

$y' + py = q$

How did we know to multiply by $\mu = e^{-2t}$?

Answer: $\mu = e^{\int p dt}$ always works.

Solving linear DEs this way is called the method of integrating factors.

Example: Solve $t\dfrac{dy}{dt} + 2y = 4t^2$

$y' + py = q$

Solving linear differential DEs with integrating factors:

1. Write DE in ‘standard form’

2. Multiply by integrating factor $\mu = e^{\int p dt}$

3. Rewrite LHS:

4. Integrate and solve for y

Deriving the method:

1. Wishful thinking:

2. Solve to find the correct expression for μ

Example 4 Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie—caused by past industrial pollution that has now been ordered to cease—is five times that of Lake Huron. If the outflow henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

Applications: Water mixtures

Example 5 A 120-gallon (gal) tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2 lb/gal of salt flows into the tank at the rate of 4 gal/min, and the well-stirred mixture flows out of the tank at the rate of 3 gal/min. How much salt does the tank contain when it is full?