MAT303: Calc IV with applications

Lecture 25 - May 5 2021

Recently: Solutions homogeneous constant coefficient systems:

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

And: Solutions to nonhomogeneous systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

Today: geometric interpretation of eigenvectors (Ch 5.3)

We will deal only with $n = 2$ case.

We've been solving systems such as

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\;\;\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned}
\tag{27}
$$

Or equivalently

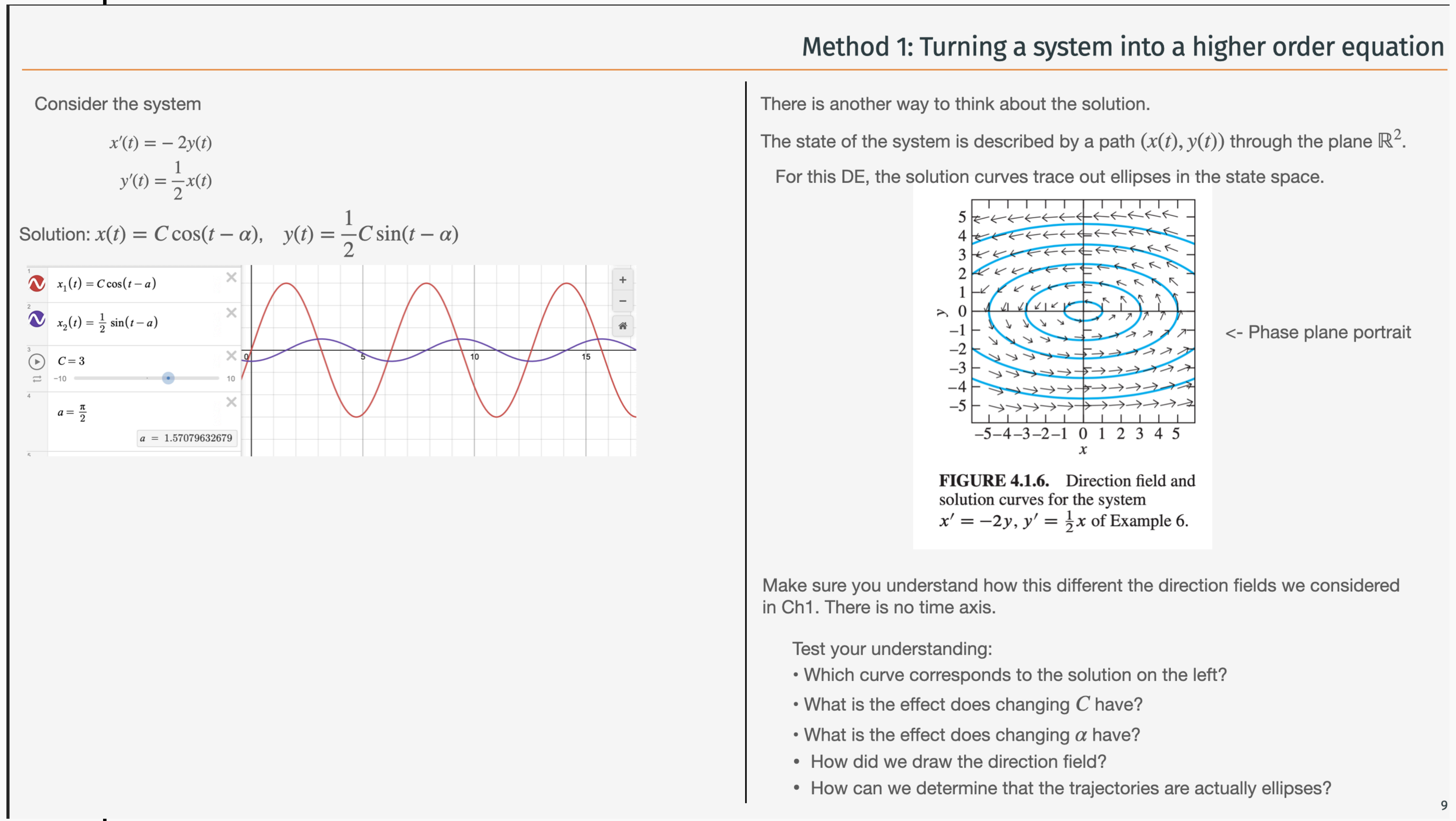
$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + f(t)$$

The solution is a collection of functions $x_1(t), \ldots, x_n(t)$, or equivalently a vector function $\mathbf{x}(t)$

Recall from your earlier education that such objects can be viewed as *parametric curves* in $\mathbb{R}^n$.

E.g. $\mathbf{x}(t) = (\cos t, \sin t)$ traces out a circle in $\mathbb{R}^2$.

So a solution to a system of DEs can be viewed as a parametric curve. See lecture 17 for more on this.



We covered how to solve these systems using the eigenvector method.

Today we'll see how the eigenvectors give us information about the solution.

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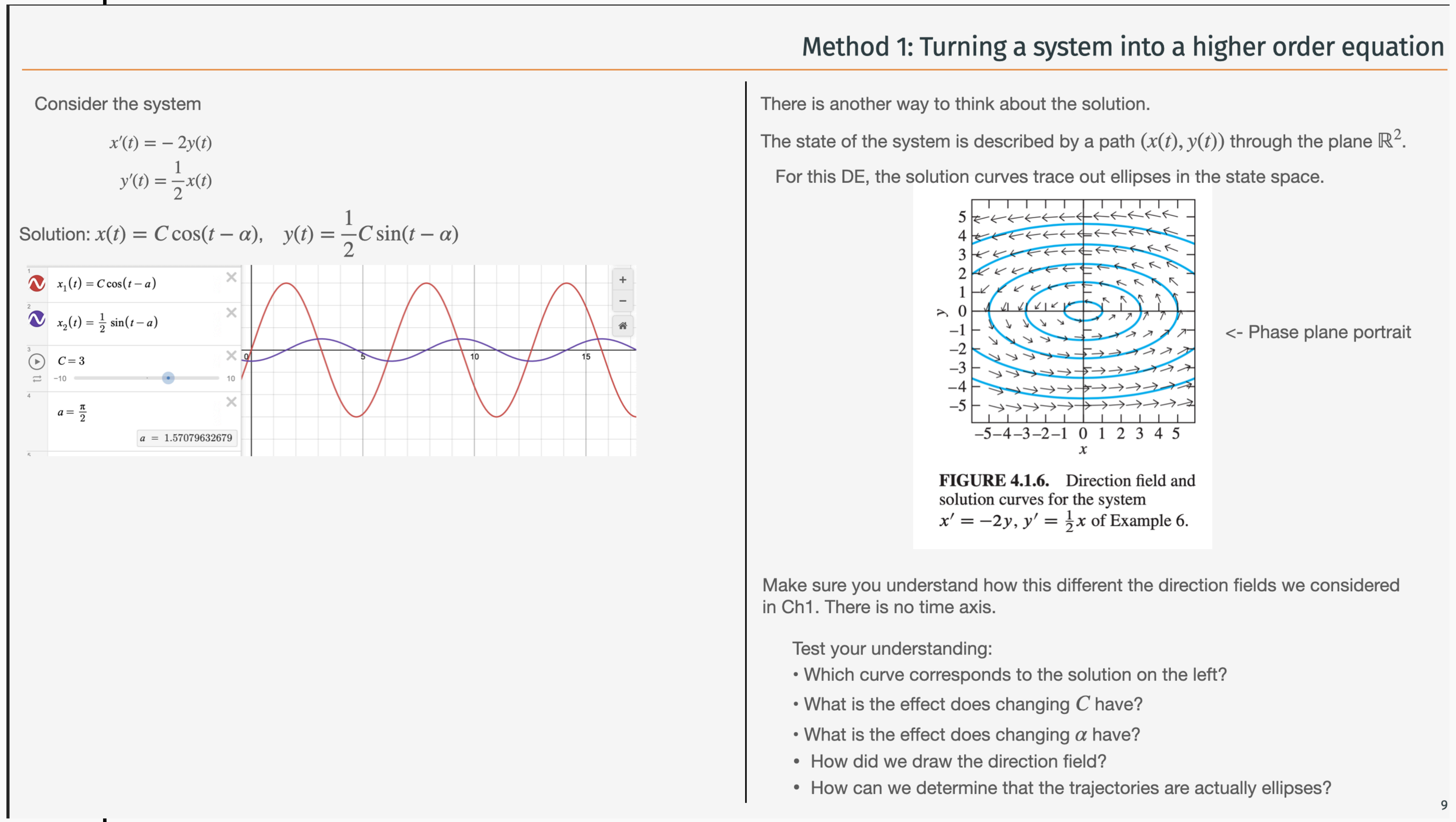

We covered how to solve these systems using the eigenvector method.

Today we'll see how the eigenvectors give us information about the solution.

Example 1: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -2, \quad \lambda_2 = 5.$

The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

General solution:

Note that $\mathbf{x}(0) = (c_1, c_2)$.

Geometrically:

The origin is called a **saddle point** for the system.

Example 2: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -14, \quad \lambda_2 = -7$

The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

General solution:

Geometrically:

- The origin is called a **sink** because all trajectories go towards the origin.
- If all the trajectories were repelled from the origin, it would be a **source.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal sink.**

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = -\begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -2 & 13 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 14, \quad \lambda_2 = 7$

The eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

This system is just the time reversal of the previous system.
Notice the eigenvalues are the same, but eigenvalues have opposite signs.

General solution:

Geometrically:

- The origin is called a **source** because all trajectories go towards the origin.
- If all the trajectories were attracted to the origin, it would be a **sink.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal source.**

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} -36 & -6 \\ 6 & 1 \end{bmatrix}$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 14, \quad \lambda_2 = 7$

The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

This system is just the time reversal of the previous system.
Notice the eigenvalues are the same, but eigenvalues have opposite signs.

General solution:

Geometrically:

- If we take the time reversal, trajectories reverse direction. Solutions are repelled from the line.

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$

The eigenvalue of $\mathbf{A}$ are $\lambda = 2$ (repeated)

A choice of linearly independent eigenvectors is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

General solution:

Geometrically:

- The origin is called a **source** because all trajectories go towards the origin.
- If all the trajectories were attracted to the origin, it would be a **sink.**
- A sink or source for which every trajectory approaches the origin tangential to a straight line is called a **node.**

So in this situation we have a **nodal source.**

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$$

The eigenvalue of $\mathbf{A}$ are $\lambda = 2$ (repeated)

There is only one eigenvector,
however we have also have a generalized eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

General solution:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2(\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}.$$

Geometrically:

In one dimension, multiplying by $\lambda \in \mathbb{R}$
can be thought of as stretching or reflection the real line:

In two dimensions, multiplying by a matrix $\mathbf{A}$
can be thought of as stretching or rotation or shearing of the plane:

E.g.

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ corresponds to rotation by $\pi/2$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$ corresponds to stretching in the x-direction

So if $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ parameterizes some curve,

then $\mathbf{A}\mathbf{x}(t)$ parameterizes a stretching/rotation/shearing of that curve.

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda = \pm\, 10i$

The eigenvectors are

$$\begin{bmatrix} 3+5i \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-5i \\ 4 \end{bmatrix}$$

General solution:

Need the following fact from linear algebra:
matrix multiplication can be geometrically interpreted as a rotation+stretch+shear.

Geometrically:

Example 3: Consider $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -17 \\ 8 & -7 \end{bmatrix}$

The eigenvalues of $\mathbf{A}$ are $\lambda = -1 \pm 10i$

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$$\begin{bmatrix} 3+5i \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-5i \\ 4 \end{bmatrix}$$

General solution:

Need the following fact from linear algebra:
matrix multiplication can be geometrically interpreted as a rotation+stretch+shear.

Geometrically:

- In this situation we have a sink.
  It's not nodal because the trajectories don't approach tangentially along a straight line.