MAT303: Calc IV with applications

Lecture 24 - May 3 2021

Recently: Solutions homogeneous constant coefficient systems:

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

Today: Solutions to nonhomogeneous systems

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

Principle of superposition:

THEOREM 4 Solutions of Nonhomogeneous Systems

Let $\mathbf{x}_p$ be a particular solution of the nonhomogeneous linear equation in (47) on an open interval I on which the functions $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be linearly independent solutions of the associated homogeneous equation on I. If $\mathbf{x}(t)$ is any solution whatsoever of Eq. (47) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t) \tag{49}$$

for all t in I.

This means we just have to find one particular solution $\mathbf{x}_p$.

Then the general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$

where $\mathbf{x}_c$ is the general homogeneous solution.

Two methods to find a particular solution:

- Method of undetermined coefficients
- Variation of parameters

Example: method of undetermined coefficients

Example 1 Find a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}. \tag{3}$$

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

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To solve

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

We assume that we have already found
the fundamental matrix to the homogeneous equation, $\mathbf{\Phi}(t)$

Therefore the general homogeneous solution is $\mathbf{x}_c = \mathbf{\Phi}(t)\mathbf{c}$

Then we guess $\mathbf{x}_p = \Phi(t)\mathbf{u}(t)$, and see what constraints on $\mathbf{u}(t)$ come up.

THEOREM 1 Variation of Parameters

If $\mathbf{\Phi}(t)$ is a fundamental matrix for the homogeneous system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on some interval where $\mathbf{P}(t)$ and $\mathbf{f}(t)$ are continuous, then a particular solution of the nonhomogeneous system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$$

is given by

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)\, dt. \tag{22}$$

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Example 4 Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} te^{-2t}, \quad \mathbf{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}. \tag{30}$$

Note: in previous lectures we found the two solutions to the homogeneous equation,

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-2t} \text{ and } \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$