MAT303: Calc IV with applications

Lecture 22 - April 26 2021

Recently:

- Eigenvalue method for $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  - Still need to finish off case of defective eigenvalues (missing solutions).

Today:

- Review matrix inverses
- Fundamental matrix solutions
  - Solve for all initial conditions 'simultaneously'.
- Matrix Exponentials as fundamental matrix solutions

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
   1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
   2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
   1. Some complications if the eigenvalue is defective (not enough eigenvectors)
5. Write down the solutions, solve for initial conditions if applicable.

Let's start with the multiplicity $k = 2$ case, it's the simplest.

Situation:

- We are trying to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- The matrix $\mathbf{A}$ has an eigenvalue λ of multiplicity 2 (repeated root)
- The eigenvalue λ is defective (only 1 linearly independent eigenvector $\mathbf{v}_1$ instead of 2).
- So we only have one solution, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$.
- Need to find another.

Solution: guess $\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ where $\mathbf{v}_2$ is unknown.

We find that the constraint on $\mathbf{v}_2$ is $(\mathbf{A} - \lambda I)^2 \mathbf{v}_2 = 0$

Note: once we find $\mathbf{v}_2$ then $\mathbf{v}_1 = (\mathbf{A} - \lambda I)\mathbf{v}_2$.

ALGORITHM Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0} \tag{16}$$

such that

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1 \tag{17}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{18}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \tag{19}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 3 Find a general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}. \tag{20}$$

Ch 5.6 Exponential Matrices and Fundamental Matrix Solutions

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - \quad y,$$

$$x(0) = 1, \quad y(0) = 1.$$

We know how to find the general solution now:

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, solve for initial conditions.

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$x(0) = 3, \quad y(0) = 2.$

We know how to find the general solution now:

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
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Insight: The IVP solution can be written as a product of a matrix and a vector.

Suppose we want to find the solution of the following initial value problem

$$x' = 4x + 2y,$$
$$y' = 3x - y,$$

$x(0) = u_1, \quad y(0) = u_2.$

We know how to find the general solution now:

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
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    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
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Insight: Just use the matrix inverse instead of solving for the coefficients.

Definition: Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be *linearly independent* solutions the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Let $\mathbf{\Phi}$ be the matrix formed by taking $\mathbf{x}_i$ as the columns.

Then $\mathbf{\Phi}$ is said to be a fundamental matrix for the system.

THEOREM 1 Fundamental Matrix Solutions

Let $\mathbf{\Phi}(t)$ be a fundamental matrix for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then the [unique] solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}\mathbf{x}_0. \tag{8}$$

The previous example, summarized with this new vocabulary:

$$\begin{aligned} x' &= 4x + 2y, \\ y' &= 3x - \ \ y, \end{aligned}$$

Takeaway: We can solve the system for *all* initial conditions, all at once. Just compute $\Phi(t)\Phi(0)^{-1}$.

It turns out that there's another, conceptually cleaner way to view fundamental solutions.

Also, this can sometimes lead to a much quicker computation.

It is inspired by the following fact:

The solution to $x' = ax$ is $x(t) = e^{at}x(0)$.

THEOREM 2 Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{26}$$

is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{27}$$

and this solution is unique.

This doesn't make sense yet, because what does $e^{\mathbf{A}t}$ mean???

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Similarly, we define

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}\,t + \mathbf{A}^2\frac{t^2}{2!} + \cdots + \mathbf{A}^n\frac{t^n}{n!} + \cdots .$$

Looks complicated because it's an infinite sum, but there are some tricks that can help us.

Example:

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

Check that the method works (all the columns are solutions to the DE):

If $A^n = 0$ for some $n,$ the matrix is said to be nilpotent.
We just saw that it is easy to compute e^{At} when A is nilpotent.

Recently:

- Eigenvalue method for $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Today:

- Review matrix inverses
- Fundamental matrix solutions
  - Solve for all initial conditions ‘simultaneously’.
- Matrix Exponentials as matrix solutions
  - Especially easy to compute when the matrix is nilpotent.

Today:

- More examples of matrix exponentials.
  - How to compute them if the matrix is not nilpotent?

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
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Using matrix exponential to solve DEs

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Find $e^{\mathbf{A}t}$
3. Solution is $\mathbf{x} = e^{\mathbf{A}t}\mathbf{x}_0$