MAT303: Calc IV with applications

Lecture 21 - April 21 2021

Last time:

- Linear independence of solutions (Finish Ch 5.1)
- Eigenvalue method (Ch 5.2)
    - Distinct real eigenvalues

Today:

- Eigenvalue method
    - Distinct complex eigenvalues (Ch 5.2)
    - Repeated eigenvalues (Ch 5.3)

Eigenvalue method

1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, use initial conditions if applicable.

Recall: **Euler's identity**

$$e^{ix} = \cos(x) + i\sin(x)$$

Recall: **Complex roots of polynomials appear in conjugate pairs**

If $p + qi$ is a root of a polynomial with real coefficients, then $p - qi$ is also a root.

Recall: **Complex conjugation**

If $z = p + qi$ then $\overline{z} = p - qi$.

Recall: **Superposition principle**

If $\mathbf{x}_1, \mathbf{x}_2$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then so is $\mathbf{x}_1 + \mathbf{x}_2$.

Another fact: **complex eigenvectors appear in pairs**

If $\mathbf{v}$ is the eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ

Then $\mathbf{v}$ is the eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\overline{\lambda}$

In other words,

If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

then $\mathbf{A}\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$

Consequence:

If $\mathbf{x}$ is a solution to the DE, $\mathrm{Re}(\mathbf{x})$ and $\mathrm{Im}(\mathbf{x})$ are also solutions.

We can use these facts deal with the case when there are complex eigenvalues.

Example 3 Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2,$$
$$\frac{dx_2}{dt} = 3x_1 + 4x_2. \quad (23)$$

- Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
- Find the eigenvalues
  - Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
  - The roots of this polynomial are the eigenvalues λ
- Find the eigenvectors corresponding to each λ

- Write down the solutions
  - If complex, take real and imaginary parts to get real solutions

Example 3 Find a general solution of the system

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1. Rewrite in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x}$
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
    - If complex, pair up conjugates and use Euler's identity to get real solutions
5. Write down the solutions

Last time: we saw that

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \text{ and } \quad \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a, \quad \mathbf{x}'(0) = b$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers c_1, $c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{35}$$

for all t in I.

Takeaway: for a $n \times n$ linear system, once we find n linearly independent solutions, we have essentially found them 'all'.

Repeated eigenvalues (Ch 5.5)

Example 1 Find a general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \mathbf{x}. \qquad (5)$$

We are always be looking for n linearly independent eigenvectors, to make sure we have found all solutions.

If an eigenvalue of multiplicity k has k linearly independent eigenvectors, it is said to be **complete.**

However, when there are repeated roots, there are sometimes there are not enough linearly independent eigenvectors…

You should always be looking for n linearly independent eigenvectors.

However, sometimes there are not enough linearly independent eigenvectors…

The following matrix only has one eigenvector.

Example 2 The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \tag{8}$$

Finding more solutions when there are defective eigenvalues

Let's start with the multiplicity $k = 2$ case, it's the simplest.

Situation:

- We are trying to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- The matrix $\mathbf{A}$ has an eigenvalue λ of multiplicity 2 (repeated root)
- The eigenvalue λ is defective (only 1 linearity independent eigenvector $\mathbf{v}_1$ instead of 2).
- So we only have one solution, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda t}$.
- Need to find another.

Solution: guess $\mathbf{x}_2 = \mathbf{v}_1 t e^{\lambda t} + \mathbf{v}_2 e^{\lambda t}$ where $\mathbf{v}_2$ is unknown.

We find that the constraint on $\mathbf{v}_2$ is $(\mathbf{A} - \lambda I)^2 \mathbf{v}_2 = 0$

Note: once we find $\mathbf{v}_2$ then $\mathbf{v}_1 = (\mathbf{A} - \lambda I)\mathbf{v}_2$.

ALGORITHM Defective Multiplicity 2 Eigenvalues

1. First find a nonzero solution $\mathbf{v}_2$ of the equation

$$(\mathbf{A} - \lambda \mathbf{I})^2 \mathbf{v}_2 = \mathbf{0} \tag{16}$$

such that

$$(\mathbf{A} - \lambda \mathbf{I}) \mathbf{v}_2 = \mathbf{v}_1 \tag{17}$$

is nonzero, and therefore is an eigenvector $\mathbf{v}_1$ associated with λ.

2. Then form the two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \tag{18}$$

and

$$\mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \tag{19}$$

of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ corresponding to λ.

Example 3 Find a general solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}. \tag{20}$$

Today:

- Eigenvalue method
  - Distinct complex eigenvalues (Ch 5.2)
    - Just use Euler's formula + superposition
  - Repeated eigenvalues (Ch 5.3)
    - If the eigenvalues are defective, must look for generalized eigenvectors
    - We only did multiplicity $k = 2$, but the same thing works for higher multiplicity.