MAT303: Calc IV with applications

Lecture 20 - April 19 2021

Last time:

- Seeing how matrix notation helps us represent systems more compactly
- Basic application of row reduction to solve for coefficients in initial value problems
- Principle of superposition

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned} \tag{27}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t).$$

Today:

- Linear independence of solutions (Finish Ch 5.1)
- Eigenvalue method (Ch 5.2)

Recall (lecture 11): linear independence of more than two functions:

DEFINITION Linear Dependence of Functions

The n functions $f_1, f_2, \ldots, f_n$ are said to be **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0 \tag{7}$$

on I; that is,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x in I.

Example:

Definition for vectors is similar:

Independence and General Solutions

Linear independence is defined in the same way for vector-valued functions as for real-valued functions (Section 3.2). The vector-valued functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are **linearly dependent** on the interval I provided that there exist constants $c_1, c_2, \ldots, c_n$ *not all zero* such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \tag{32}$$

for all t in I. Otherwise, they are **linearly independent**. Equivalently, they are

Example:

Another way to check linear independence is through the Wronksian, see textbook.

Last time: we saw that

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \mathbf{x} = \mathbf{P}\mathbf{x}.$$

Has two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \text{ and } \tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

And we can take linear combinations to get new solutions:

$$\mathbf{x} = c_1 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

We could choose c_1 and c_2 to match initial conditions $\mathbf{x}(0) = a, \quad \mathbf{x}'(0) = b$

Compare this to the following single second order equation:

$$y'' - 2y' + y = 0$$

We can easily find two solutions:

$$y = e^t \text{ and } y = te^t$$

And we can take linear combinations to get new solutions:

$$y = c_1 e^t + c_2 te^t$$

We could choose c_1 and c_2 to match initial conditions $y(0) = a, \quad y'(0) = b$

- We know that once we find two linearly independent solutions, all other solutions are linear combinations.

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THEOREM 3 General Solutions of Homogeneous Systems

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I, where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I, then there exist numbers c_1, $c_2, \ldots, c_n$ such that

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{35}$$

for all t in I.

Takeaway: for a $n \times n$ linear system, once we find n linearly independent solutions, we have essentially found them 'all'.

Eigenvalue method (Ch 5.2)

We wish to find solutions $(x_1, \ldots, x_n)$ to the system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

We know from Ch 5.1 that we can write this more compactly as

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

- Make the following guess: $\mathbf{x} = \mathbf{v}e^{\lambda t}$
- Substitute into DE, giving
- Therefore $\mathbf{v}$ and λ solve $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

We've simplified the problem to an algebraic problem.

How to solve this algebraic problem? $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$

This is called an eigenvalue/eigenvector problem.

The λ solutions are called *eigenvalues* of $\mathbf{A}$

The $\mathbf{v}$ solutions are called *eigenvectors* of $\mathbf{A}$

Two methods:

- Treat as undetermined system
- Use characteristic polynomial

Example 1 Find a general solution of the system

$$x_1' = 4x_1 + 2x_2,$$
$$x_2' = 3x_1 - x_2.$$

Solution The matrix form of the system in (11) is

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}. \quad (12)$$

1. Guess that solution is of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Substitute into (12).

2. Get an eigenvalue problem.

3. Solve the eigenvalue problem.

4. Use eigenvectors to write down solution to the DE.

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The $n \times n$ identity matrix is the matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

It is useful because for all matrices $\mathbf{A}$, we have $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$.

To solve the eigenvalue problem $\mathbf{Av} = \lambda\mathbf{v}$, we can first find the eigenvalues by solving the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Using this new method for the previous example:

1. Make the matrix $\mathbf{A} - \lambda\mathbf{I}$

2. Find determinant of $\mathbf{A} - \lambda\mathbf{I}$

3. Solve for roots λ.

Works exactly the same for larger systems:

$$
\begin{aligned}
x_1' &= -k_1 x_1, \\
x_2' &= k_1 x_1 - k_2 x_2, \\
x_3' &= \phantom{-k_1 x_1 - {}} k_2 x_2 - k_3 x_3,
\end{aligned}
$$

1. Rewrite in matrix form
2. Use the guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, get the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$
3. Find the eigenvalues
    1. Form the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
    2. The roots of this polynomial are the eigenvalues λ
4. Find the eigenvectors corresponding to each λ
5. Write down the solutions, use initial conditions if applicable.

$$
\mathbf{x}'(t) = \begin{bmatrix} -0.5 & 0.0 & 0.0 \\ 0.5 & -0.25 & 0.0 \\ 0.0 & 0.25 & -0.2 \end{bmatrix} \mathbf{x},
$$

Works exactly the same for larger systems:

$$\begin{aligned} x_1' &= -k_1 x_1, \\ x_2' &= \quad k_1 x_1 - k_2 x_2, \\ x_3' &= \qquad\qquad\; k_2 x_2 - k_3 x_3, \end{aligned}$$

1. Rewrite in matrix form
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Today:
- How to reduce the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$
  to the eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.
- How to solve the eigenvalue problem for some eigenvalues and eigenvectors.

Actually, sometimes we won't get n real eigenvectors.
There could be missing solutions, or some them could be complex.

We'll talk about how to deal with those cases next time. (Ch 5.2, 5.5).