MAT303: Calc IV with applications

Lecture 19 - April 14 2021

Recently started looking at:

- Systems of differential equations (analogous to systems of algebraic equations)

Today:

- Seeing how matrix notation helps us represent systems more compactly
- Basic application of row reduction to solve for coefficients in initial value problems

The goal of today is to understand why…

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t),\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t),\\
x_3' &= p_{31}(t)x_1 + p_{32}(t)x_2 + \cdots + p_{3n}(t)x_n + f_3(t),\\
&\;\;\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t).
\end{aligned} \tag{27}
$$

…can be written as:

$$
\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t).
$$

At its core, a $n \times m$ matrix is a "table of numbers":

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}. \tag{1}$$

Adding and subtracting matrices is natural:

You can only do this if the dimensions match:

If you haven't taken linear algebra, matrix multiplication is strange:

You can only multiply matrices if their dimensions match:

In many respects, matrices behave like numbers.

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}, \qquad \mathbf{A} - \mathbf{A} = \mathbf{0}; \tag{6}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{(commutativity)}; \tag{7}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad \text{(associativity)}; \tag{8}$$

$$\begin{aligned} c(\mathbf{A} + \mathbf{B}) &= c\mathbf{A} + c\mathbf{B}, \\ (c + d)\mathbf{A} &= c\mathbf{A} + d\mathbf{A}. \end{aligned} \qquad \text{(distributivity)} \tag{9}$$

$A(B + C) = AB + AC$ distributivity

However, **multiplication is not commutative.**

Consider the system of equations

$$x_1' = 4x_1 - 3x_2,$$
$$x_2' = 6x_1 - 7x_2$$

We can rewrite this as

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\mathbf{x} = \mathbf{P}\mathbf{x}.$$

We've achieved our initial goal: we've rewritten our system as a matrix equation.

Explanation:

Example usage of matrices and vectors:

$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ is a solution to $\dfrac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x}$.

$\mathbf{x} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ is another solution to $\dfrac{d\mathbf{x}}{dt} = \mathbf{P}\mathbf{x}$.

Consider the system of equations

$$x_1' = 4x_1 - 3x_2,$$
$$x_2' = 6x_1 - 7x_2$$

Two solutions:

$$x_1 = 3e^{2t}, \quad x_2 = 2e^{2t}$$ and

$$\tilde{x}_1 = e^{-5t}, \quad \tilde{x}_2 = 3e^{-5t}$$

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\mathbf{x} = \mathbf{P}\mathbf{x}.$$

Two solutions:

$$\mathbf{x} = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$ and

$$\tilde{\mathbf{x}} = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x}, \tag{29}$$

THEOREM 1 Principle of Superposition

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be n solutions of the homogeneous linear equation in (29) on the open interval I. If $c_1, c_2, \ldots, c_n$ are constants, then the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t) \tag{31}$$

is also a solution of Eq. (29) on I.

Notice how it is conceptually much cleaner,
because we don't have to wrote out the whole matrices and vectors
We're just using these properties, which say that we can treat
the matrices and vectors as one "unit".

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}, \qquad \mathbf{A} - \mathbf{A} = \mathbf{0}; \tag{6}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \text{(commutativity)}; \tag{7}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad \text{(associativity)}; \tag{8}$$

$$\left.\begin{aligned} c(\mathbf{A} + \mathbf{B}) &= c\mathbf{A} + c\mathbf{B}, \\ (c + d)\mathbf{A} &= c\mathbf{A} + d\mathbf{A}. \end{aligned}\right\} \qquad \text{(distributivity)} \tag{9}$$

$A(B + C) = AB + AC$ distributivity

Row reduction

For the uninitiated it can be daunting to solve a system of equations such as this:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{43}$$

However, we can all agree that it is easy to solve systems that look like this:

$$\begin{aligned} \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \cdots + \bar{a}_{1n}x_n &= \bar{b}_1, \\ \bar{a}_{22}x_2 + \cdots + \bar{a}_{2n}x_n &= \bar{b}_2, \\ &\vdots \\ \bar{a}_{nn}x_n &= \bar{b}_n \end{aligned}$$

But it’s actually easy to convert the top form to the bottom form.

Just be systematic about eliminating the leftmost variables.

Example:

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0, \\ 2c_1 \qquad\quad - 2c_3 &= 2, \\ c_1 - c_2 + c_3 &= 6 \end{aligned}$$

Search “gaussian elimination” for more examples.

Example:

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0, \\ 2c_1 \qquad\quad - 2c_3 &= 2, \\ c_1 - c_2 + c_3 &= 6 \end{aligned}$$

Notice that the variables c_1, c_2, c_3. don't actually contain any information. We can write the computation on the left like this:

Now we can understand the following problem:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

are solutions of the equation

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\mathbf{x}. \qquad (34)$$

Use these solutions to solve the initial value problem

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}.$$

Solution:

- By the principle of superposition, any linear combination of the given solutions is another solution.

- The initial conditions become:

- We can use row reduction to solve this:

Using matrices:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \mathbf{x}_2(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

are solutions of the equation

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}. \tag{34}$$

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}.$$

Without using matrices:

Recently started looking at:

- Systems of differential equations (analogous to systems of algebraic equations)

Today:

- Understanding matrices and especially matrix multiplication allows us to represent equations in a more compact form
- Row reduction is a good way so systematically solve linear algebraic systems