MAT303: Calc IV with applications

Lecture 18 - April 12 2021

So far this class:

- Looking at single differential equations

Rest of the class:

- Systems of differential equations (analogous to systems of algebraic equations)

Consider the system

$$
\begin{aligned}
x' &= 4x - 3y \\
y' &= 6x - 7y \\
x(0) = 2, \quad y(0) &= -1
\end{aligned}
$$

Analogously to algebraic systems, we can try eliminating one of the variables:

Consider the constant coefficient linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad (1)$$

We can rewrite this as

$$Ly = 0$$

where $L = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_0$ is an **operator**

where $D = \dfrac{d}{dx}$ is the **derivative operator**

Examples of operator notation:

"Multiplication" of differential operators

$L_1 L_2 f$ means $L_1(L_2 f)$

Constant coefficient polynomial differential operators are commutative:

Consider the system

$$\begin{aligned} x' &= 4x - 3y \\ y' &= 6x - 7y \\ x(0) = 2, &\quad y(0) = -1 \end{aligned}$$

We will solve this more systemically, using differential operators.

Rewrite this as:

$$\begin{aligned} -4x + x' + 3y &= 0 \\ -6x + 7y + y' &= 0 \\ x(0) = 2, &\quad y(0) = -1 \end{aligned}$$

Rewrite this as:

$$\begin{aligned} (-4 + D)x + 3y &= 0 \\ -6x + (7 + D)y &= 0 \\ x(0) = 2, &\quad y(0) = -1 \end{aligned}$$

Rewrite this as:

$$\begin{aligned} L_1x + L_2y &= 0 \\ L_3x + L_4y &= 0 \\ x(0) = 2, &\quad y(0) = -1 \end{aligned}$$

Where $L_1 = D - 4, \quad L_2 = 3, \quad L_3 = -6, \quad L_4 = D + 7$

Operate on top equation by L_3 and bottom equation by L_1 :

$$\begin{aligned} L_3L_1x + L_3L_2y &= 0 \\ L_1L_3x + L_1L_4y &= 0 \\ x(0) = 2, &\quad y(0) = -1 \end{aligned}$$

Subtract top equation from bottom:

$$L_1L_4y - L_3L_2y = 0$$

This is a single variable equation which we can solve.

In the previous example, the key milestone was rewriting the equation in the form

$$L_1x + L_2y = 0$$
$$L_3x + L_4y = 0$$

In general, whenever we can write our system in the form

$$L_1x + L_2y = f_1(t)$$
$$L_3x + L_4y = f_2(t)$$

we can do the same trick:

- Operate on top equation by L_3 and bottom equation by L_1 :

$$L_3L_1x + L_3L_2y = L_3f_1(t)$$
$$L_1L_3x + L_1L_4y = L_1f_2(t)$$

Subtract top equation from bottom:

$$L_1L_4y - L_3L_2y = L_1f_2(t) - L_3f_1(t)$$

This is a single variable equation which we can solve for y.

- Operate on top equation by L_4 and bottom equation by L_2 :

$$L_4L_1x + L_4L_2y = L_4f_1(t)$$
$$L_2L_3x + L_2L_4y = L_2f_2(t)$$

Subtract bottom equation from top:

$$L_4L_1x - L_2L_3x = L_4f_2(t) - L_2f_1(t)$$

This is a single variable equation which we can solve for x.

Summary:

The solution to

$$L_1x + L_2y = f_1(t)$$
$$L_3x + L_4y = f_2(t)$$

is given by solving

$$(L_4L_1 - L_2L_3)x = L_4f_2(t) - L_2f_1(t)$$

and

$$(L_1L_4 - L_3L_2)y = L_1f_2(t) - L_3f_1(t)$$

Example: suppose want to solve the system

$$\begin{aligned} x' - 4x + 3y &= t \\ -6x + y' + 7y &= 0 \end{aligned}$$

Rewrite in terms of of differential operators:

According to what we just did:

Summary:

The solution to

$$\begin{aligned} L_1 x + L_2 y &= f_1(t) \\ L_3 x + L_4 y &= f_2(t) \end{aligned}$$

is given by solving

$$(L_4 L_1 - L_2 L_3)x = L_4 f_2(t) - L_2 f_1(t)$$

and

$$(L_1 L_4 - L_3 L_2)y = L_1 f_2(t) - L_3 f_1(t)$$

To solve 2x2 linear equation:

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

We can use Cramer's rule:

$$x = \frac{pd - qb}{ad - bc}, \quad y = \frac{aq - cp}{ad - bc}$$

Summary (to solve 2x2 first order system):

The solution to

$$\begin{aligned} L_1x + L_2y &= f_1(t) \\ L_3x + L_4y &= f_2(t) \end{aligned}$$

is given by solving

$$(L_4L_1 - L_2L_3)x = L_4f_2(t) - L_2f_1(t)$$

and

$$(L_1L_4 - L_3L_2)y = L_1f_2(t) - L_3f_1(t)$$

Today:
- Solving systems of first order equations by elimination
- Using polynomial differential operators

Next time: Chapter 5.1
- Better methods for larger systems