MAT303: Calc IV with applications

Lecture 16 - March 31 2021

Recently: Initial Value Problems

- Looking at $my'' + cy' + ky = f(t)$

  where we are solving for $y(t)$, $y(0) = a, y'(0) = b$

Today: Endpoint problems (Ch 3.8)

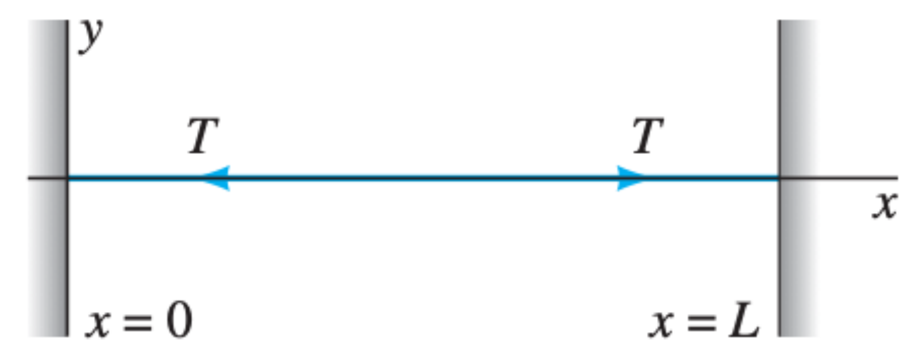
- $y'' + p(x)y' + q(x)y = 0$, where we are solving for $y(x)$
- Key difference: boundaries conditions of the form $y(a) = \cdots, y(b) = \cdots$

The shape of a jump rope being twirled satisfies the following equation:

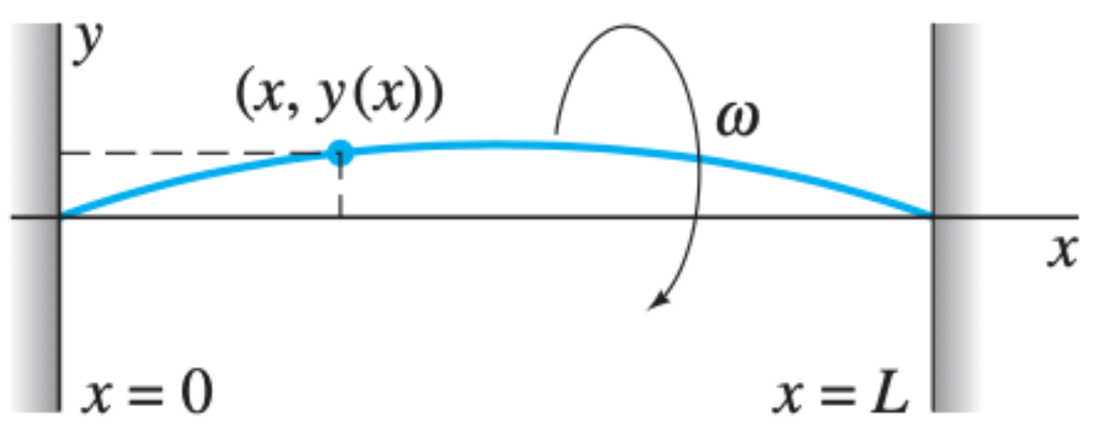
$$Ty'' + \rho\omega^2 y = 0$$
$$y(0) = 0$$
$$y(L) = 0$$

Where

- ω is angular frequency (constant)
- ρ is density fo the string (mass per unit length, constant)
- L is the length of the string (constant)
- y is the radial deviation from the center
- $0 \leq x \leq L$ is the location along the string we are measuring the deviation
- T is the tension force due to spring (constant)



Equilibrium position



Whirling string
(b)

Explanation of the equation (not a full derivation):

- Net y component of tension force is Ty''
  - y component of force is proportional to y'
  - So net y component of force is prop. to y''

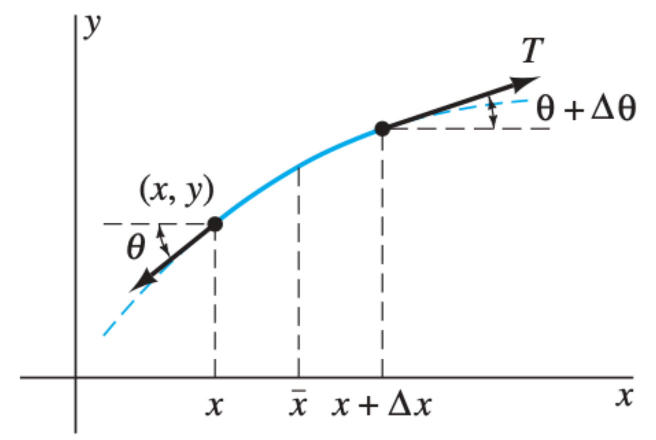


FIGURE 3.8.7. Forces on a short segment of the whirling string.

- Centripetal force is $-\rho\omega^2 y$
  - Further away from center -> bigger circle -> greater force

We see that it is natural to have boundary conditions at different points 0 and L, instead of at the same point.

Instead of
$$\begin{aligned} Ty'' + \rho\omega^2 y &= 0 \\ y(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

Look at $y'' + 3y = 0; \quad y(0) = 0, \quad y(\pi) = 0$ (For simplicity)

Solution:

Look at $y'' + 4y = 0; \quad y(0) = 0, \quad y(\pi) = 0$ (For simplicity)

Solution:

We see that $y'' + \lambda y = 0; \quad y(0) = 0, \quad y(\pi) = 0$

only has solutions for certain λ.

Let's find out when there are solutions (i.e. let's solve the "eigenvalue problem"):

Contrast to initial value problem, where every choice of λ has a solution:

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y'(0) = 0$$

Recall from linear algebra:

Let A be a matrix (linear operator)

If we wish to find vectors v and numbers λ such that

$$Av = \lambda v$$

This is called an eigenvalue problem.

Solutions might only exists for certain λ and certain v.

If (λ, v) is a solution, λ is called an eigenvalue and v is called an eigenvector.

Note: if v is an eigenvector, so is any constant multiple of v.

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(\pi) = 0$$

Solve the eigenvalue problem

(For simplicity)

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y'(\pi) = 0$$

Solution:

The shape of a jump rope being twirled satisfies the following equation:

$$\begin{aligned} Ty'' + \rho\omega^2 y &= 0 \\ y(0) &= 0 \\ y(L) &= 0 \end{aligned}$$

ρ, T, L are constants.

Conclusion: You can only rotate a jump rope at a certain angular frequencies:

Even though the equations

$$y'' + p(x)y' + \lambda y = 0, \quad y(0) = 0, \quad y'(0) = 0 \quad (1)$$

and

$$y'' + p(x)y' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0 \quad (2)$$

look superficially similar, the solutions have very different properties.

- For (1), existence and uniqueness of IVPs usually applies, no matter what λ is.
- For (2), sometime there is no solutions y, otherwise there are infinitely many, depending on λ.
  - Alternatively, we may think of (2) as a joint equation for λ,y.

  To solve (2), we have to solve for both λ and y.