MAT303: Calc IV with applications

Lecture 14 - March 24 2021

Last time:

- Physical interpretation of

  $my'' + cy' + ky = f(t)$

  in terms of mass-spring systems

- When $f(t) = 0$, saw that there are 3 regimes, depending on whether $c < 4km$:
  - Underdamped
  - Critically damped
  - Overdamped

Today:

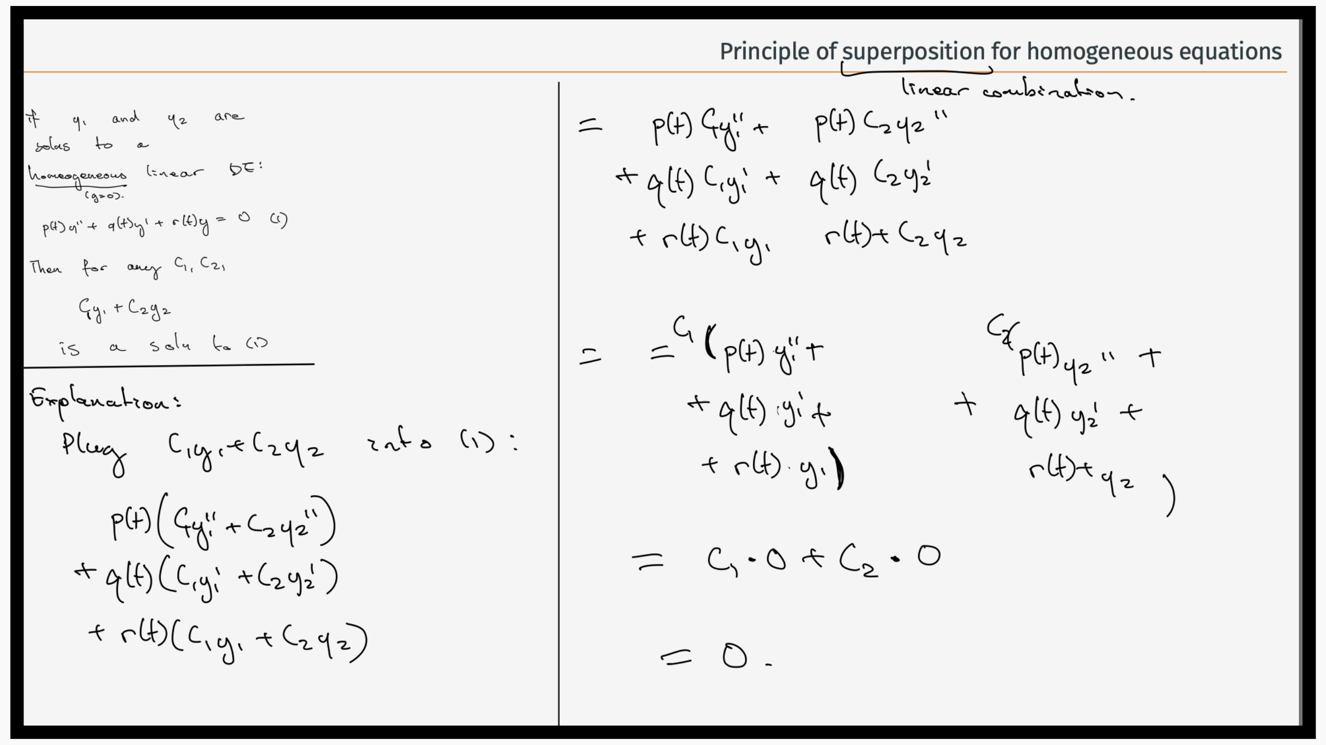
- Solving nonhomogeneous linear DEs

Principle of superposition for homogeneous linear equations:

If $y_1$and y_2 are solutions to

$$ay'' + by' + cy = 0$$

Then $Ay_1 + By_2$ is also a solution.



Principle of superposition for non-homogeneous linear equations:

If $y_1$is a solution to

$$ay'' + by' + cy = f_1(x)$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(x)$$

Then $Ay_1 + By_2$ is a solution to

$$ay'' + by' + cy = Af_1(x) + Bf_2(x)$$

How do we deal with external force, e.g.

$$y'' - 4y = 2e^{3x} \,?$$

Recall (lecture 11)

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (2) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \ldots, y_n$ be linearly independent solutions of the associated homogeneous equation in (3). If Y is any solution whatsoever of Eq. (2) on I, then there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) \qquad (16)$$

for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$

  where y_h is a solution to the homogeneous version of the equation.

Solution:

How do we deal with external force, e.g.

$$y'' + 3y' + 4y = 3x + 2?$$

Recall (lecture 11)

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for all x in I.

Roughly speaking:

- All solutions are of the form $Y(x) = y_c + y_p$ where y_c is a solution to the homogeneous version of the equation.

Solution:

How do we deal with external force, e.g.

$$3y'' + 3y' - 2y = 2\cos x?$$

Recall (lecture 11)

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Roughly speaking:

- All solutions are of the form $Y(x) = y_h + y_p$

  where y_h is a solution to the homogeneous version of the equation.

Solution:

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \ldots$

RULE 1 Method of Undetermined Coefficients

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

To solve constant coefficient linear differential equation $ay'' + by' + cy = f(x)$,

1. Find the homogeneous (i.e. complementary) solutions y_c
2. Check that f and its derivatives don't satisfy the homogeneous equation
3. Determine y_p by guessing $y_p =$ linear combination of f and its derivatives, and solve for coefficients
4. General solution is $y_c + y_p$

Examples:

- $y'' - 4y = 2e^{3x}$?
- $y'' + 3y' + 4y = 3x + 2$
- $3y'' + 3y' - 2y = 2\cos x$
- $y'' - 4y = 2e^{2x}$

How to deal with this situation?

$$y'' - 4y = 2e^{2x}.$$

Want to take $y_p = Ae^{2x}$ as trial solution, but it's a solution to the homogeneous equation.

Solution: Multiply trial solution by x.

In general:

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \ldots$

RULE 2 Method of Undetermined Coefficients

If the function $f(x)$ is of either form in (14), take as the trial solution

$$y_p(x) = x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx], \quad (15)$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (15) by substituting y_p into the nonhomogeneous equation.

Translation: keep multiplying trial solution by x until it's no longer a solution to the homogeneous equation.

Consider the equation

$$y'' + y = \tan(x).$$

Homogeneous solutions:

$$y = A\cos(x) + B\sin(x)$$

Unfortunately, the sequence $f, f', f'', \ldots$ has infinitely many linearly independent terms.

$$\sec^2 x, \quad 2\sec^2 x \tan x, \quad 4\sec^2 x \tan^2 x + 2\sec^4 x, \quad \ldots$$

I.e. The vector space spanned by f and its derivatives is infinite dimensional.

We can't use method of undetermined coefficients.

Variation of parameters

Want to solve $y'' + Py' + Qy = f(x)$

Key ideas:

- Guess $y_p = u_1 y_1 + u_2 y_2$ where y_1, y_2 are the homogeneous solutions.

  Plug this into $L[y_p] = 0$ and see what this tells us about u_1, u_2.
- We are free to make one more additional constraint to make our computation easier

THEOREM 1 Variation of Parameters

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1 y_1(x) + c_2 y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx, \qquad (33)$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Consider a system of two linear equations in two variables.

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0; \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, D \neq 0.$$

Back to our example:

$y'' + y = \tan(x).$

Homogeneous solutions:

$y_1 = \cos(x),\ y_2 = \sin(x)$

Assume that only finitely many linearly independent functions appears in the sequence $f, f', f'', f''', \ldots$

RULE 1 Method of Undetermined Coefficients

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Need to explain why $y_p = C_0 f + C_1 f' + C_2 f'' + \cdots + C_n f^{(n)}$

Proof sketch:

1. Let V be the vector space spanned by $f, f', f'', f''', \ldots$.

   We are assuming that V is finite dimensional.
2. Then L is a linear operator $V \to V$.
3. By the rank nullity theorem from linear algebra, one of the two possibilities always holds:
   - $Lg = 0$ for some $g \in V$, i.e. $\dim(\ker L) > 0$.
   - $Lg = h$ always has a solution $g \in V$.
4. Therefore, if we assume that the first doesn't hold, then the second must hold.

What about rule 2?

RULE 2 Method of Undetermined Coefficients

If the function $f(x)$ is of either form in (14), take as the trial solution

$$y_p(x) = x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx + (B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx], \quad (15)$$

where s is the smallest nonnegative integer such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (15) by substituting y_p into the nonhomogeneous equation.

Need to explain why $y_p = x^s(C_0 f + C_1 f' + C_2 f'' + \cdots + C_n f^{(n)})$

Proof sketch:

1. Let V be the vector space spanned by $f, f', f'', f''', \ldots$.

   We are assuming that V is finite dimensional.
2. Then L is a linear operator $V \to V$, and $\dim(\ker L) > 0$.
3. Consider the vector space $W = \{x^s g : g \in V\}$.
   - Check: If s is small enough, $L : W \to V$.
   - Check: As s increases, $\dim(\ker L)$ decreases
4. Therefore, if s is just right, $\dim(\ker L) = 0$ and so $Lg = h$ always a solution $g \in W$.