MAT303: Calc IV with applications

Lecture 12 - March 17 2021

Recently:

- Second order linear differential equations (Ch 3.1)
  - Homogeneous equations
  - Principle of superposition
  - Special case: constant coefficients
    - Different cases depending on number of real roots
  - Existence and uniqueness
  - Linear independence, and general solutions

$$y'' + p(x)y' + q(x)y = 0$$

Last time:

- Higher order linear differential equations (Ch 3.2)
- Mostly the same as second order linear differential equations
  - Difference: linear independence is more subtle
- Non-homogeneous equations

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0. \quad (3)$$

Today:

- Constant coefficient higher order linear differential equations
- Similar to $n = 2$ case, but we will introduce some new tools:
  - Linear Differential Operators
- Spend some more time on Euler's identity $e^{ix} = \cos(x) + i\sin(x)$

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

Consider the constant coefficient linear equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

We can rewrite this as

$$Ly = 0$$

where $L = a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \cdots + a_0$ is an **operator**

where $D = \dfrac{d}{dx}$ is the **derivative operator**

Examples of operator notation:

Example: Find a solution of the differential equation $(D^2 + 5D + 6)y = 0.$

Let $p(x)$ be a polynomial of degree n. Then p has exactly n roots (possibly complex, possibly repeated).

Complex roots always appear in conjugate pairs $a + ib$ and $a - ib$

- It may be hard to find the roots (there is no “quadratic formula” for $n \geq 5$
- But we know by the theorem that they do exist

Let $i = \sqrt{-1}$, so $i^2 = -1$.

Euler's identity:

$$e^{ix} = \cos x + i\sin(x)$$

Recall power series:

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Example 4.1.3. Suppose that $i = \sqrt{-1}$ is the imaginary unit. Then,

$$\begin{aligned} e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} &= \sum_{k=0}^{\infty} i^{2k} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i^{2k+1} \frac{x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i(-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= \cos x + i \sin x \end{aligned}$$

Let's try to solve

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

Use our only trick: guess $y = e^{rx}$.

If the $r_1, \ldots, r_n$ are distinct, the $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent. (Can compute Wronskian by induction).

So we have found n linearly independent solutions,
so by the theory of last lecture,
all solutions are of the form

$$y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$$

If the $r_1, \ldots, r_n$ not distinct, the $e^{r_1 x}, \ldots, e^{r_n x}$ are dependent.

By the theory of last lecture, we are missing some solutions.

We deduce the missing solutions by using linear differential operators.

Suppose we have 2 distinct roots, r_1 and r_2, where r_2 is repeated k times.

Notice that

THEOREM 2 Repeated Roots

If the characteristic equation in (3) has a repeated root r of multiplicity k, then the part of a general solution of the differential equation in (1) corresponding to r is of the form

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}. \tag{14}$$

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Example: Find a general solution of the fifth-order differential equation

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

If the r_1 and r_2 are complex conjugate roots, $r_1, r_2 = a \pm ib$,

We get the two contributions $e^{r_1 x}$ and $e^{r_2 x}$ to the general solution.

We would prefer real solutions. We can get these by using Euler's formula.

THEOREM 3 Complex Roots

If the characteristic equation in (3) has an unrepeated pair of complex conjugate roots $a \pm bi$ (with $b \neq 0$), then the corresponding part of a general solution of Eq. (1) has the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx). \tag{21}$$

Example: Find a general solution of $y^{(4)} + 4y = 0$

Today:

- Constant coefficient higher order linear differential equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \quad (1)$$

  - Characteristic equation arises from substituting $y = e^{rx}$
    - If roots of characteristic equation are distinct, general solution is

$$y = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$$

    - For a repeated root r of order k, the contribution to the general solution is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{rx}$$

    - For a non-repeated conjugate pair of complex roots $a \pm bi$, the contribution to the general solution is

$$y = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

    - For repeated pairs of complex roots of order k, the contribution is

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{ax} \cos(bx) + (d_1 + d_2 x + d_3 x^2 + \cdots + d_k x^{k-1}) e^{ax} \sin(bx)$$