MAT303: Calc IV with applications

Lecture 9 - March 8 2021

Last time:

- Second order linear differential equations (Ch 3.1)
    - Homogeneous equations
    - Principle of superposition
    - Constant coefficient case
        - Real roots
        - Imaginary roots

Today:

- Second order linear differential equations (Ch 3.1)
    - Existence and uniqueness
    - Linear independence, and general solutions

- Consider the functions $y_1 = e^x$ and $y_2 = 3e^x$.

Then $Ay_1 + By_2 = Ce^x$.

Even though there are seemingly two parameters A and B, it is really a one parameter family.

- Contrast with the situation $y_1 = e^x$ and $y_2 = 3e^{2x}$.

Now $y = Ay_1 + By_2$ is genuinely a two-parameter family.

Linear independence of functions:
Two functions are linearly independent
if they are not multiples of each other

Last time: finding the general solution to

$$y'' + 5y' + 6y = 0$$

- Substituted in $y = e^{rt}$ as a guess
- Lead to the equation $r^2 + 5r + 6 = 0$
- Therefore $r = -2, -3$.
- So $y_1 = e^{-2t}$ and $y_2 = e^{-3t}$ are 'solutions'.
- By superposition, $y = Ae^{-2t} + Be^{-3t}$ is a solution too (for any choice of A and B)

Now suppose that we impose an initial condition $y(0) = 2$ and $y'(0) = 3$.

Things to notice:

- We needed 2 constraints to completely determine the solution.

Questions:

- Is this the only solution to the IVP?
- Why are there solutions at all? Will we always have solutions?
  - What happens if we change the initial conditions?

Another motivating example:

$$y'' - 2y' + y = 0$$
$$y(0) = 2$$
$$y'(0) = 3$$

We see that if the characteristic has repeated roots we run into problems:

Motivation: How do we know that solutions to differential equations exist? How do we know that there's only one solution?

THEOREM 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{8}$$

has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \tag{11}$$

Previous example:

$$y'' + 5y' + 6y = 0$$
$$y(0) = 2$$
$$y'(0) = 3$$

An example where the theorem does not apply:

$$y'' + x^{-1}y' + 6y = 0$$
$$y(0) = 2$$
$$y'(0) = 3$$

An example where the theorem does apply:

$$y'' + x^{-1}y' + 6y = 0$$
$$y(1) = 2$$
$$y'(1) = 3$$

Back to our example:

$$y'' + 5y' + 6y = 0$$

- By superposition, $y = Ae^{-2t} + Be^{-3t}$ is a solution (for any choice of A and B)

However, we still don't "know" that all the solutions are of the form $y = Ae^{-2t} + Be^{-3t}$.

For that, we need this theorem:

THEOREM 4 General Solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation (Eq. (9))

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq. (9) on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

THEOREM 2 Existence and Uniqueness for Linear Equations

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has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1. \tag{11}$$

We see now that it is important to know whether two functions are linearly independent.

Here is an easy way to check if two functions are linearly independent.

THEOREM 3 Wronskians of Solutions

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation (Eq. (9))

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.

(a) If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.

(b) If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Wronskian:

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Example: $y_1 = e^{-x}, \quad y_2 = xe^{-x}$

Example: $y_1 = e^{-x}, \quad y_2 = 4e^{-x}$

- For a second order linear differential equation

  $y'' + p(x)y' + q(x)y = 0$

  - If $p(x)$ and $q(x)$ are nice, for every choice of initial values $y(a) = \alpha$ and $y'(a) = \beta$, a solution will exist.
  - If y_1 and y_2 are a pair of linearly independent solutions, then every solution is of the form $C_1y_1 + C_2y_2$.
  - We can check if y_1 and y_2 are linearly independent by computing the Wronskian.

Example:

$y'' - 4y = 0$

We have two solutions $y_1 = e^{2x}$ and $y_2 = e^{-2x}$.

We also have solutions $w_1 = e^{2x} + e^{-2x}$
and $w_2 = e^{2x} - e^{-2x}$.

Conclusion: all solutions are of the form $C_1e^{-2x} + C_2e^{2x}$

Conclusion: all solutions are of the form $C_1(e^{-2x} + e^{2x}) + C_2(e^{-2x} - e^{2x})$

- For a second order linear differential equation

  $y'' + p(x)y' + q(x)y = 0$

  - If $p(x)$ and $q(x)$ are nice, for every choice of initial values $y(a) = \alpha$ and $y'(a) = \beta$, a solution will exist.
  - If y_1 and y_2 are a pair of linearly independent solutions, then every solution is of the form $C_1y_1 + C_2y_2$.
  - We can check if y_1 and y_2 are linearly independent by computing the Wronskian.

Example:

$$y'' - 2y' + y = 0$$

We have two solutions $y_1 = e^x$ and $y_2 = xe^x$.

Conclusion: all solutions are of the form $C_1e^x + C_2xe^x$

(This always happens when the characteristic equation has repeated roots)
See Theorem 6 in textbook.

Last time: finding the general solution to

$$y'' + 2y' + 2y = 0$$

- Substituted in $y = e^{rt}$ as a guess
- Lead to the equation $r^2 + 2r + 2 = 0$
- Therefore $r = -1 \pm i$.
- So $y_1 = e^{(-1+i)t}$ and $y_2 = e^{(-1-i)t}$ are 'solutions'.

Last time:

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  - Homogeneous equations
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Today:

- Second order linear differential equations (Ch 3.1)
  - Constant coefficient case
    - Imaginary roots
    - Repeated roots

Let $i = \sqrt{-1}$, so $i^2 = -1$.

Euler's identity:

$$e^{ix} = \cos x + i\sin(x)$$

Recall power series:

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$
- $\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

- $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
- $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Example 4.1.3. Suppose that $i = \sqrt{-1}$ is the imaginary unit. Then,

$$\begin{aligned} e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} &= \sum_{k=0}^{\infty} i^{2k} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i^{2k+1} \frac{x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} i(-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= \cos x + i \sin x \end{aligned}$$