MAT200: Logic Language and Proof

Lecture 4 - February 15 2021

HW1 is graded on gradescope.

Lecture 3:

Example 1: For positive real numbers a, b,

$$a < b \implies a^2 < b^2$$

Example 4:
$(-1) \cdot 3 = -3$

Example:
For any a real, $a \cdot 0 = 0$.
If a is a real number, then $a \cdot 0 = 0$

Example 6:
Prove:
if a and b are even, then $a \cdot b$ is even.

* Proof By cases
* Proof By contradiction
* Proof By contrapositive.

Example 7:
Prove:
if a and b are even, then $a + b$ is even.

Written up proof:

1. Suppose a and b are numbers.
2. Then $a + b = 2\left(\dfrac{a+b}{2}\right)$.
3. Thus we have shown that $a + b$ is of the form $2q$ where q is integer.

Where is the mistake?

Lesson:
1. The correctness of a proof is independent of the correctness of the statement.
2. When pinpointing an error, be precise. Stick to things that are actually logically wrong.

Example 2: If $a = 1$ or $a = 2$ then $a^2 - 3a + 2 = 0$.

if $a=1$, then

$$a^2 - 3a + 2 = (1)^2 - 3(1) + 2$$
$$= 1 - 3 + 2$$
$$= 0$$

if $a=2$, then

$$a^2 - 3a + 2 = (2)^2 - 3(2) + 2$$
$$= 4 - 6 + 2$$
$$= 0.$$

QED.

Example:

101 is odd.

1. Is it true?
    1. Make sure you actually understand the statement.
        1. Formal understanding
        2. Informal understanding
    2. Look up all relevant definitions
    3. Try some examples.
2. Why is it true? (Rough idea).
3. Translate your idea a nicely written proof.
    1. Try to make your proof sound proofs you read from the textbook(s).
        1. If it sounds awkward, re-write it.
        2. Your proof should consist of grammatically correct English sentences.
    2. Make sure every leap of logic can be explained.

Instead: n is odd iff n is not even.

2. Want: 101 is not even.

i.e. if q is integer, $101 \neq 2q$

$2(50) \neq 101$ $\quad 2(3) \neq 101$

$2(51) \neq 101$ $\quad 2(4) \neq 101$

Infinitely many integers, how can we be sure we tried them all.

3. We need to prove that 101 is not even.

That is, we need to show that

If q is integer, $2q \neq 101$.

Let q be an integer.

By trichotomy, either $q \leq 50$ or $q \geq 51$.

- If $q \leq 50$, then $2q \leq 100 < 101$.
- If $q \geq 51$, then $2q \geq 102 > 101$.

In either case, $2q \neq 101$.

2 cases:

if $q \leq 50$: $2q \leq 100$

if $q \geq 51$: $2q \geq 102$

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Proof Ideas

Definitions

Written proof

Axioms

Example:

There do not exist integers m and n such that

$$14m + 20n = 101.$$

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1. Examples:

$m=2, n=3$ $\quad 14\times 2 + 20 \times 3 \neq 101$

$m=1, n=1$ $\quad 14\times 1 + 20 \times 1 \neq 101$

2.

Given	Goal
m, n integer	$14m+20n \neq 101$

Given	Goal
m, n integer $14m+20n = 101$	Contradiction

If $14m+20n=101$,
then $2(7m+10n) = 101$.
So 101 is even. This contradicts the fact that 101 is odd.

- Suppose for contradiction that m, n are integers such that $14m + 20n = 101$.
- Then $2(7m + 10n) = 101$.
- Thus $101 = 2q$ for some integer q, so 101 is even.
- This contradicts the fact that 101 is odd.

Why it works: if you start with some assumptions, and they can be used to get a contradiction, one of assumptions was false.

You should use facts that you've proved.

Example:

101 is odd.

Depends on:

1 is not even.

We used: Unity law:

* $1 > 0$ Proof:
* $1 < 2$.

Exercise: Prove these.

$101 = 2q$

$1 = 2q - 100$ subtract 100

$1 = 2(q - 50)$ factorize.

We've shown $1 = 2l$, l is integer.

i.e. 1 is even.

We need to prove that 101 is not even.

Suppose for contradiction that 101 is even.

Then $101 = 2q$

Then $1 = 101 - 100 = 2(q - 50)$.

If $q - 50 \leq 0$ then $1 \leq 0$, a contradiction.

If $q - 50 \geq 1$ then $1 \geq 2$, also a contradiction.

So 101 cannot be even.

1 is eve[n]

contradict[ion]

if $q \geq 51$, $1 \geq 2(51-50) = 2$: contradiction

if $q \leq 50$, $1 \leq 0$: contradiction

Trying to prove $1 > 0$.

Only relevant axiom: Unity law: $1 \cdot a = a$.

So $1 = 1 \cdot 1$.

So just need to prove

$x^2 > 0$.

Exercise. (Use 3.1.2).

2.3.1

Know

$\cancel{x}^2 > 0$.

$x^2 = 0$.

$x^2 \geq 0$

$x > 0$.

Recall:

$$P \implies Q \quad \text{is equivalent to} \quad \neg Q \implies \neg P$$

This can used to transform statements into something easier to prove.

Assum
$a > b$

Example:

$$\left[ac \leq bc \implies c \leq 0 \right]$$

1. Is it true?
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      1. Formal understanding
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2. Why is it true? (Rough idea).
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Axioms 3.1.2

(i) Trichotomy law. *For each pair of real numbers a and b, one and only one of the three possibilities $a < b$, $a = b$, $a > b$ is true.*

(ii) Addition law. *For real numbers a, band c,*

$$a < b \Leftrightarrow a + c < b + c.$$

(iii) Multiplication law. *For real numbers a, band c,*

$$a < b \Leftrightarrow ac < bc \quad \text{if } c > 0,$$
$$a < b \Leftrightarrow ac > bc \quad \text{if } c < 0.$$

(iv) Transitive law. *For real numbers a, band c,*

$$a < b \text{ and } b < c \Rightarrow a < c.$$

Contrapositive: Assume $a > b$.

$$c > 0 \implies ac > bc$$

We prove the contrapositive: if $c > 0$ then $ac > bc$.

This is axiom 3.1.2 ii) in the textbook. QED.

Assume the opposite (negation) of what you want to prove.

Recall:

$P \implies Q$ is equivalent to $\neg Q \implies \neg P$

This can used to transform statements into something easier to prove.

Example:

If n^2 is odd then n is odd.

2) Proof by contradiction

Given	Want
n^2 is odd	Contradiction
n is even	

$\Downarrow$

$n = 2q$, q integer

$\Downarrow$

$n^2 = 4q^2$

$\Downarrow$

$n^2 = 2(2q^2) \Rightarrow n^2$ is even.

Contradiction.

Suppose n^2 is odd, and suppose for contradiction that n is even. Then $n = 2q$, for some integer q. So $n^2 = 2(2q^2)$ is even! This contradicts assumption that n^2 is odd.

1. Examples:

n	n
1	1
2	4
3	9
6	36
7	49

Definitions:

n odd $\Leftrightarrow$ n is not even.

2. Contrapositive: n is even $\Rightarrow$ n^2 is even

Given	Want
n is even	n^2 is even

$\Downarrow$

$n = 2q$, q integer.

$\Rightarrow n^2 = 4q^2 \Rightarrow n^2 = 2(2q^2).$

We prove the contrapositive: if n is even then n^2 is even.
Suppose n is even, then n=2q where q is an integer.
Then $n^2 = 4q^2 = 2(2q^2)$, so we have shown that n^2 is even. QED.

Example:

$ab = 0 \implies a = 0$ or $b = 0$

Try doing this with contradiction / contrapositive.

Equivalently:

$ab=0$ and $a \neq 0 \Rightarrow b=0$

Generally:

$P \Rightarrow Q$ or R

$\equiv$ P and $\neg Q \Rightarrow R$.

statement is equivalent to:

$ab=0$ and $a \neq 0 \Rightarrow b=0$.

Given	Want
$ab=0$ $a \neq 0$	$b=0$

(vii) Division. *If $a \neq 0$, then equation $ax = b$ has the unique solution $x = b/a = ba^{-1}$. (This means that we can cancel: $ax_1 = ax_2 \Rightarrow x_1 = x_2$ so long as $a \neq 0$.)*

It suffices to prove that

$ab = 0$ and $a \neq 0 \implies$ b=0.

Suppose a, b are real numbers such that $ab = 0$, and suppose $a \neq 0$.

By division axiom, $a^{-1} \cdot a = 1$.

Multiplying both sides by a^{-1} yields $\underbrace{a^{-1}a}_{1}b = a^{-1} \cdot 0$.

The left hand side is equal to b, by unity

The right hand side is equal to 0, because for all x, $x \cdot 0 = 0$.

So b=0 as desired. QED.

- Proof by contradiction
- Proof by cases
- Proof by contrapositive
- Proving an ‘or’ statement.

Warning: many proofs do not fit neatly into one of these categories.

Especially try to avoid proof by contradiction (usually proof by contrapositive is better).